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Instationary Generalized Stokes Equations in Partially Periodic Domains

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Abstract. We consider an instationary generalized Stokes system with nonhomogeneous divergence data under a periodic condition in only some directions. The problem is set in the whole space, the half space or in (after an identification of the periodic directions with a torus) bounded domains with sufficiently regular boundary. We show unique solvability for all times in Muckenhoupt weighted Lebesgue spaces. The divergence condition is dealt with by analyzing the associated reduced Stokes system and in particular by showing maximal regularity of the partially periodic reduced Stokes operator.

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1. Introduction

Consider the partially periodic instationary generalized Stokes problem

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} = f & \text{in } (0,T)\times\Omega, \\ \operatorname{div} u = g & \text{in } (0,T)\times\Omega, \\ u|_{\partial\Omega} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u|_{t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $u : (0,T)\times\Omega \to \mathbb{R}^n$ is the fluid velocity and $\mathfrak{p} : (0,T)\times\Omega \to \mathbb{R}$ is the pressure. Here, $T \in (0,\infty]$ and Ω is a domain in $G := \mathbb{R}^{n_1} \times \mathbb{T}_L^{n_2}$ with $\mathbb{T}_L := \mathbb{R}/L\mathbb{Z}$, $L > 0$ and $n := n_1 + n_2 \geq 2$. The topology and differentiable structure on G are the canonical ones inherited from $\mathbb{R}^n$, so that (1.1) governs a flow which is periodic of length L in the direction of the variables $y := (x_{n_1+1}, \ldots, x_n)$. Such partially periodic models are relevant in mathematical fluid mechanics, for example in the analysis of flows in spiraling tubes or layer-like domains with periodic boundary conditions.

Assumption 1.1. We want to consider problem (1.1) in

- the partially periodic whole space $G := \{(x', y) \mid x' := (x_1, \ldots, x_{n_1}) \in \mathbb{R}^{n_1}, y := (x_{n_1+1}, \ldots, x_n) \in \mathbb{T}_L^{n_2}\}$;
- the partially periodic half space $G_+ := \{x \in G \mid x_1 > 0\}$;
- bounded partially periodic $C^{1,1}$-domains, that is bounded, open and connected $\Omega \subset G$, where the boundary can be described locally (after a possible rotation of the coordinate system) as the graph of a $C^{1,1}$-function.

The nonperiodic case $n_2 = 0$ has been extensively investigated in the literature in a variety of domains. Bothe and Prüss [3] considered general instationary Stokes systems in bounded and exterior domains for Dirichlet, Neumann and Navier boundary conditions. Unique solvability of (1.1) in Sobolev spaces for

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Birkhäuser

a large class of domains including bounded and exterior domains, asymptotically flat layers, infinite cylinders, perturbed half spaces and aperture domains was obtained by Abels [2], where also variable viscosity and mixed boundary conditions are admitted. The main idea of Abels [2] is to use maximal L^p regularity of some associated Stokes operator. We want to follow this train of thought and establish a theory which enables us to show a corresponding regularity result for the partially periodic reduced Stokes operator, i.e, for all $n_2 \in \{0, \ldots, n-1\}$.

For $n_2 > 0$, there are only very few results in the literature. In the case of a homogeneous divergence condition, i.e, $g = 0$, there are early results by Iooss [13] in the L^2 framework. In L^p Sobolev spaces, problem (1.1) has been treated by Denk and Nau [8,17] in the case of a straight cylinder and by the author [20,22,23] in the whole space case $\Omega = G$. In particular, Theorem 3.5 in [23] shows that the partially periodic Stokes operator admits maximal L^p regularity in $\mathrm{L}^q_{\omega,\sigma}(G)$ for all $q \in (1,\infty)$ and all $\omega \in A_q(G)$ for $n \geq 3$ with an A_q-consistent estimate. Here, $A_q(G)$ is the class of Muckenhoupt weights $\omega \in A_q(\mathbb{R}^n)$ which are periodic of length L with respect to the variables $y = (x_{n_1+1}, \ldots, x_n)$, cf. [22, Proposition 2]. Recall that a nonnegative $\omega \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ is in the Muckenhoupt class $A_q(\mathbb{R}^n)$, if

$$\mathcal{A}_q(\omega)^{\frac{1}{q}} := \sup_B \frac{1}{|B|} \|\omega\|^{\frac{1}{q}}_{\mathrm{L}^1(B)} \|\omega'\|^{\frac{1}{q'}}_{\mathrm{L}^1(B)} < \infty,$$

where $\omega' := \omega^{-q'/q}$ and the supremum runs over all balls $B \subset \mathbb{R}^n$. The reason to include weighted spaces lies in an extrapolation theorem in the spirit of García-Cuervo and Rubio de Francia [12], which roughly states that uniform bounds in weighted spaces immediately extend to $\mathcal{R}$-bounds. More precisely, the following proposition can be found in [20, Theorem 2]. Here, we call a constant $c = c(\omega)$ that depends on Muckenhoupt weights A_q-consistent, if for each $d > 0$ we have

$$\sup\{c(\omega) : \omega \text{ is an } A_q(G)\text{-weight with } \mathcal{A}_q(\omega) < d\} < \infty.$$

Proposition 1.2. *Suppose that $r, q \in (1,\infty)$, $\omega \in A_q(G)$ and that $\Omega \subset G$ is measurable. Moreover, assume that $\mathcal{T}$ is a family of linear operators such that for all $\nu \in A_r(G)$ there is an A_r-consistent constant $c_r = c_r(\nu) > 0$ with*

$$\|Tf\|_{L^r_\nu(\Omega)} \leq c_r \|f\|_{L^r_\nu(\Omega)}$$

for all $f \in L^r_\nu(\Omega)$ and all $T \in \mathcal{T}$. Then every $T \in \mathcal{T}$ can be extended to $L^q_\omega(\Omega)$ and $\mathcal{T}$ is $\mathcal{R}$-bounded with an A_q-consistent $\mathcal{R}$-bound c_q.

Since $\mathcal{R}$-boundedness of solutions to the corresponding resolvent equations is connected to maximal L^p regularity via the Theorem of Weis (see Proposition 2.2 below), Proposition 1.2 suggests that key in understanding problem (1.1) is a thorough investigation in weighted spaces of the resolvent problem

$$\begin{cases} \lambda u - \Delta u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda \in \Sigma_\vartheta := \{\lambda \in \mathbb{C} : |\arg \lambda| < \vartheta, \lambda \neq 0\}$, $\vartheta \in (0,\pi)$. In particular, we aim at obtaining *a priori* estimates which are A_q-consistent.

The purpose of the present paper is threefold:

- extend the results in [2] to the partially periodic case and to weighted spaces,
- extend the results on the corresponding resolvent equations in [22] to all dimensions $n \geq 2$ and to domains with boundary, and
- extend the results on the maximal L^p regularity of the partially periodic Stokes operator in [23] to non-homogeneous divergence data g and to domains with boundary.

Our main results are stated in the following two theorems. Here, the space of initial values is defined as the real interpolation space $B^{2-2/p}_{q,p,\omega}(\Omega) := \big(\mathrm{L}^q_\omega(\Omega), W^{2,q}_\omega(\Omega) \cap W^{1,q}_{0,\omega}(\Omega)\big)_{1-1/p,p}$, which can be regarded as a partially periodic, weighted space of Besov type. The precise definition of the respective function spaces can be found in Sect. 3.

Theorem 1.3. *Let $n \geq 2$ and let Ω be as in Assumption 1.1. Assume $\mathsf{T} \in (0,\infty)$, $p, q \in (1,\infty)$ and $\omega \in A_q(G)$. Then there is an A_q-consistent constant $c = c(n,p,q,\omega,\Omega,\mathsf{T}) > 0$ such that for all $T \in (0,\mathsf{T}]$, all $f \in \mathrm{L}^p(0,T;\mathrm{L}^q_\omega(\Omega)^n)$, all $g \in \mathrm{L}^p(0,T;W^{1,q}_\omega(\Omega))$ with $\partial_t g \in \mathrm{L}^p(0,T;\widehat{W}^{-1,q}_{0,\omega}(\Omega))$ and all $u_0 \in B^{2-2/p}_{p,q,\omega}(\Omega)^n$ satisfying the compatibility condition*

$$\operatorname{div} u_0 = g|_{t=0} \quad \text{in } \widehat{W}^{-1,q}_{0,\omega}(\Omega),$$

there is a unique $(u,\mathfrak{p}) \in \big(\mathrm{L}^p(0,T;W^{2,q}_\omega(\Omega)^n) \cap W^{1,p}(0,T;\mathrm{L}^q_\omega(\Omega)^n)\big) \times \mathrm{L}^p(0,T;\widehat{W}^{1,q}_\omega(\Omega))$ solving (1.1) and it holds the estimate

$$\|u, \partial_t u, \nabla^2 u, \nabla\mathfrak{p}\|_{\mathrm{L}^p(\mathrm{L}^q_\omega)} \leq c\left(\|f,\nabla g\|_{\mathrm{L}^p(\mathrm{L}^q_\omega)} + \|\partial_t g\|_{\mathrm{L}^p(\widehat{W}^{-1,q}_{0,\omega})} + \|u_0\|_{B^{2-\frac{2}{p}}_{p,q,\omega}}\right). \tag{1.3}$$

If Ω is bounded, the assertion remains true for $\mathsf{T} = \infty$.

As explained above, the key ingredient in the proof of Theorem 1.3 is the following result.

Theorem 1.4. *Let $n \geq 2$ and Ω be as in Assumption 1.1. Let $q, q_i \in (1,\infty)$, $\omega \in A_q(G)$, $\omega_i \in A_{q_i}(G)$, $i = 1,2$, $\vartheta \in (0,\pi)$, $\lambda \in \Sigma_\vartheta$ (for bounded Ω also $\lambda = 0$ is permitted).*

(i) *For each $f \in \mathrm{L}^q_\omega(\Omega)^n$ and $g \in W^{1,q}_\omega(\Omega) \cap \widehat{W}^{-1,q}_{0,\omega}(\Omega)$ there is a unique solution $(u,\mathfrak{p}) \in W^{2,q}_\omega(\Omega)^n \times \widehat{W}^{1,q}_\omega(\Omega)$ to (1.2). This solution satisfies*

$$\|\lambda u, \nabla^2 u, \nabla\mathfrak{p}\|_{\mathrm{L}^q_\omega(\Omega)} \leq c\left(\|f,\nabla g\|_{\mathrm{L}^q_\omega(\Omega)} + |\lambda|\|g\|_{\widehat{W}^{-1,q}_{0,\omega}(\Omega)}\right), \tag{1.4}$$

where $c = c(n,q,\omega,\vartheta,\Omega) > 0$ is A_q-consistent. In the case of a bounded domain, the term $\|\nabla^2 u\|_{\mathrm{L}^q_\omega(\Omega)}$ on the left-hand side may be replaced by $\|u\|_{W^{2,q}_\omega(\Omega)}$.

(ii) *If $f \in \mathrm{L}^{q_1}_{\omega_1}(\Omega)^n \cap \mathrm{L}^{q_2}_{\omega_2}(\Omega)^n$ and both $g \in W^{1,q_1}_{\omega_1}(\Omega) \cap W^{1,q_2}_{\omega_2}(\Omega)$ and $g \in \widehat{W}^{-1,q_1}_{0,\omega_1}(\Omega) \cap \widehat{W}^{-1,q_2}_{0,\omega_2}(\Omega)$, then the unique solution $(u,\mathfrak{p}) \in W^{2,q_1}_{\omega_1}(\Omega)^n \times \widehat{W}^{1,q_1}_{\omega_1}(\Omega)$ satisfies the regularity $(u,\mathfrak{p}) \in W^{2,q_2}_{\omega_2}(\Omega)^n \times \widehat{W}^{1,q_2}_{\omega_2}(\Omega)$.*

Remark 1.5. Note that for bounded partially periodic $C^{1,1}$-domains, a homogeneous flux condition is built in into our functional analytic setting: Consider for example a periodic cylinder $\Omega := \mathbb{D} \times \mathbb{R}/L\mathbb{Z}$, where $\mathbb{D} \subset \mathbb{R}^{n-1}$ is the unit disc. Then for the corresponding pressure $\mathfrak{p} \in \widehat{W}^{1,q}_\omega(\Omega)$ from Theorem 1.4 it does not only hold that $\nabla\mathfrak{p} \in \mathrm{L}^q_\omega(\Omega)^n$, but also that $\mathfrak{p}$ itself is periodic in the sense $\mathfrak{p}|_{x_n \downarrow 0} = \mathfrak{p}|_{x_n \uparrow L}$. Therefore, the pressure drop within one periodic cell is zero, which results in a homogeneous flux condition.

The paper is structured as follows: Firstly, in Sect. 2 we prove that Theorem 1.3 can be deduced from Theorem 1.4 by arguments similar to the ones in [1,2]. The notation and basic results on weighted Lebesgue and Sobolev spaces defined over domains in the group G are provided in Sect. 3. The main part of this paper are Sects. 4–6, which are devoted to establishing Theorem 1.4. In Sect. 4, the case $\Omega = G$ is treated. Sections 5 and 6 are concerned with Theorem 1.4 in the cases of the half space and bounded periodic $C^{1,1}$-domains, respectively. It should be pointed out that the treatment of bounded domains in Sect. 6 is very different in style compared to the Sects. 4 and 5. In fact, since bounded domains have a finite measure and are relatively compact, standard localization techniques can be applied to show the corresponding regularity estimates, which reduces a large part of the problem to the nonperiodic case. Observe that in doing so, it is also *necessary* to use non-periodic results, as one is rotating the coordinate system during the process of localization, which is not compatible with having distinguished directions of periodicity.

Finally, in "Appendix", we give a construction of the Helmholtz decomposition in weighted partially periodic spaces.

2. Proof of Theorem 1.3

Let us show that Theorem 1.4 indeed implies Theorem 1.3. Therefore, consider the reduced partially periodic Stokes equations

$$\begin{cases} \lambda u - \Delta u + \nabla P u = f_r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $P : W^{2,q}_{\omega}(\Omega)^n \cap W^{1,q}_{0,\omega}(\Omega)^n \to \widehat{W}^{1,q}_{\omega}(\Omega)$ gives the unique solution to

$$(\nabla P u, \nabla\varphi) = (\Delta u, \nabla\varphi) - (\nabla \operatorname{div} u, \nabla\varphi), \quad \varphi \in \widehat{W}^{1,q'}_{\omega'}(\Omega)$$

and $f_r := f - \nabla\mathfrak{p}_r$, where $\mathfrak{p}_r \in \widehat{W}^{1,q}_{\omega}(\Omega)$ is the unique solution to

$$(\nabla\mathfrak{p}_r, \nabla\varphi) = (f, \nabla\varphi) + (\nabla g, \nabla\varphi) + \lambda[g, \varphi], \quad \varphi \in \widehat{W}^{1,q'}_{\omega'}(\Omega).$$

Observe that the unique solvability follows from the Helmholtz projection, more precisely from Lemma 7.1.

Lemma 2.1. *Let $n \geq 2$ and Ω be as in Assumption 1.1. Let $q \in (1, \infty)$, $\omega \in A_q(G)$, $\vartheta \in (0, \pi)$ and $\lambda \in \Sigma_\vartheta$. For every $f_r \in \mathrm{L}^q_\omega(\Omega)^n$ there is a unique solution $u \in W^{2,q}_\omega(\Omega)^n$ to (2.1). Moreover, there is an A_q-consistent $c = c(n, q, \vartheta, \omega, \Omega) > 0$ such that*

$$\|\lambda u, \nabla^2 u\|_{\mathrm{L}^q_\omega(\Omega)} \leq c\|f_r\|_{\mathrm{L}^q_\omega(\Omega)}. \tag{2.2}$$

If Ω is a bounded domain, also $\lambda = 0$ is permitted.

Proof. By Theorem 1.4, there is a solution $(u, \mathfrak{p}) \in W^{2,q}_\omega(\Omega)^n \times \widehat{W}^{1,q}_\omega(\Omega)$ to (1.2) with data (f_r, g), where $g \in W^{1,q}_\omega(\Omega)$ with $\lambda g \in \widehat{W}^{-1,q}_{0,\omega}(\Omega)$ is the unique solution to

$$\lambda(g, \varphi) + (\nabla g, \nabla\varphi) = (f_r, \nabla\varphi), \quad \varphi \in \widehat{W}^{1,q'}_{\omega'}(\Omega),$$

which exists due to Lemma 7.1, and where also $\lambda = 0$ is allowed in the case of a bounded domain. Then it is immediate that

$$(\nabla\mathfrak{p}, \nabla\varphi) = (\Delta u, \varphi) - (\nabla \operatorname{div} u, \varphi), \quad \varphi \in \widehat{W}^{1,q'}_{\omega'}(\Omega)$$

and hence $\mathfrak{p} = Pu$. Thus u solves (2.1) and

$$\|\lambda u, \nabla^2 u\|_{\mathrm{L}^q_\omega(\Omega)} \leq c\left(\|f_r, \nabla g\|_{\mathrm{L}^q_\omega(\Omega)} + |\lambda|\|g\|_{\widehat{W}^{-1,q}_{0,\omega}(\Omega)}\right) \leq c\|f_r\|_{\mathrm{L}^q_\omega(\Omega)}.$$

□

For $q \in (1, \infty)$ and $\omega \in A_q(G)$, we define the partially periodic reduced Stokes operator $A^{\mathrm{red}}_{q,\omega}$ on $\mathrm{L}^q_\omega(\Omega)$ via

$$D(A^{\mathrm{red}}_{q,\omega}) := W^{2,q}_\omega(\Omega)^n \cap W^{1,q}_{0,\omega}(\Omega)^n$$
$$A^{\mathrm{red}}_{q,\omega} u := -\Delta u + \nabla P u.$$

We want to show that $A^{\mathrm{red}}_{q,\omega}$ admits maximal L^p regularity. Here, a generator $-A$ of a bounded analytic semi-group on a Banach space X is said to admit maximal L^p-regularity, if for all $f \in \mathrm{L}^p(0, \infty; X)$ and $u_0 \in (X, D(A))_{1-1/p,p}$, the mild solution to

$$u_t + Au = f, \quad u(0) = u_0$$

is a.e. $D(A)$-valued, a.e. differentiable with values in X and such that both u_t and Au belong to $\mathrm{L}^p(0, \infty; X)$. Recall the Theorem of Weis [25].

Proposition 2.2. *Let $p \in (1, \infty)$ and assume that $-A$ is the generator of a bounded analytic semi-group in an UMD space X. Then A admits maximal L^p-regularity if and only if the operator family $\{it(it + A)^{-1} : t \in \mathbb{R}, t \neq 0\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(X)$.*

Note that for $q \in (1,\infty)$, all closed subspaces of $\mathrm{L}^q(\Omega,\mu)$ are UMD spaces [5].

Theorem 2.3. *Let $n \geq 2$ and Ω be as in Assumption 1.1. Let $p, q \in (1,\infty)$ and $\omega \in A_q(G)$. The partially periodic reduced Stokes operator $A^{\mathrm{red}}_{q,\omega}$ admits maximal L^p regularity. In particular, for $\mathsf{T} \in (0,\infty)$, there is an A_q-consistent $c = c(n,p,q,\omega,\Omega) > 0$ such that for every $T \in (0,\mathsf{T}]$, every $f_r \in \mathrm{L}^p(0,T;\mathrm{L}^q_\omega(\Omega))$ and every $u_0 \in B^{2-2/p}_{p,q,\omega}(\Omega)$ there is a unique solution $u \in \mathrm{L}^p(0,T;D(A^{\mathrm{red}}_{q,\omega})) \cap W^{1,p}(0,T;\mathrm{L}^q_\omega(\Omega))$ to the abstract Cauchy problem*

$$\begin{aligned} \partial_t u + A^{\mathrm{red}}_{q,\omega} u &= f_r, \\ u(0) &= u_0, \end{aligned}$$

and it holds the estimate

$$\|u, \partial_t u, \nabla^2 u\|_{\mathrm{L}^p(0,T;\mathrm{L}^q_\omega(\Omega))} \leq c \left(\|f_r\|_{\mathrm{L}^p(0,T;\mathrm{L}^q_\omega(\Omega))} + \|u_0\|_{B^{2-\frac{2}{p}}_{p,q,\omega}(\Omega)} \right).$$

If Ω is bounded, also $\mathsf{T} = \infty$ is permitted.

Proof. Lemma 2.1 shows that the family of operators $\{\lambda(\lambda + A^{\mathrm{red}}_{q,\omega})^{-1} \mid \lambda \in i\mathbb{R}, \lambda \neq 0\}$ is uniformly bounded in $\mathcal{L}(\mathrm{L}^q_\omega(\Omega))$. Proposition 1.2 shows that it is even $\mathcal{R}$-bounded. Thus, the Theorem of Weis applies. Note that in [23, Theorem 2.11], an A_q-consistent version of Weis' Theorem has been given, which justifies the claimed A_q-consistency of the bound c. Since $A^{\mathrm{red}}_{q,\omega}$ is invertible on bounded domains by Lemma 2.1, the additional remark also follows from the Theorem of Weis. $\square$

We can now give the proof of Theorem 1.3. Uniqueness of solutions to (1.1) follows directly from Theorem 2.3, since for $f = 0$ and $g = 0$ we have $A^{\mathrm{red}}_{q,\omega} u = -\Delta u + \nabla \mathfrak{p}$. Hence, we can concentrate on the existence part. Let f, g, and u_0 be given as in the theorem and define for almost all $t \in (0,T)$ the pressure $\mathfrak{p}_r(t)$ via

$$(\nabla \mathfrak{p}_r(t), \nabla\varphi) = (f(t), \nabla\varphi) + (\nabla g(t), \nabla\varphi) + [\partial_t g(t), \varphi], \quad \varphi \in \widehat{W}^{1,q'}_{\omega'}(\Omega).$$

By the assumptions on f and g, we see $\nabla \mathfrak{p}_r \in \mathrm{L}^p(0,T;\mathrm{L}^q_\omega(\Omega))$.

Then $(u,\mathfrak{p})$ is the desired solution to (1.1), where u is obtained by Theorem 2.3 with $f_r := f - \nabla\mathfrak{p}_r$, and where $\mathfrak{p} := Pu + \mathfrak{p}_r$. Indeed, it remains only to verify $\operatorname{div} u = g$.

By the definition of u, we have for almost all $t \in (0,T)$ and all $\varphi \in \widehat{W}^{1,q'}_{\omega'}(\Omega)$

$$-[\partial_t \operatorname{div} u(t), \varphi] - (\nabla \operatorname{div} u(t), \nabla\varphi) = (f_r(t), \nabla\varphi) = -[\partial_t g(t), \varphi] - (\nabla g(t), \nabla\varphi).$$

Thus, if we define $w := \operatorname{div} u - g \in \mathrm{L}^p(0,T;W^{1,q}_\omega(\Omega))$, we have $\partial_t w \in \mathrm{L}^p(0,T;\widehat{W}^{-1,q}_{0,\omega}(\Omega))$,

$$\int_0^T [\partial_t w(t), \varphi(t)]\, \mathrm{d}t + (\nabla w, \nabla\varphi) = 0, \quad \varphi \in \mathrm{L}^{p'}(0,T;W^{1,q'}_{\omega'}(\Omega)), \tag{2.3}$$

and $w|_{t=0} = 0$ by the compatibility condition on u_0 and g. Let $\varphi \in \mathrm{L}^{p'}(0,T;\widehat{W}^{1,q'}_{\omega'}(\Omega))$ be fixed but arbitrary and denote by $v \in \mathrm{L}^{p'}(0,T;W^{2,q'}_{\omega'}(\Omega))$ the solution obtained from Theorem 2.3 with right-hand side $\nabla\varphi$ and $v_0 = 0$. Then

$$-(\nabla\varphi, \nabla\psi) = \int_0^T [\partial_t \operatorname{div} v(t), \psi(t)]\, \mathrm{d}t + (\nabla \operatorname{div} v, \nabla\psi)$$

for all $\psi \in \mathrm{L}^p(0,T;\widehat{W}^{1,q}_\omega(\Omega))$, in particular for $\tilde{w}(t,x) := w(T-t,x)$. Using $v(0) = \tilde{w}(T) = 0$, we obtain with (2.3)

$$\begin{aligned} -(\nabla\varphi, \nabla\tilde{w}) &= \int_0^T [\partial_t \operatorname{div} v(t), \tilde{w}(t)]\, \mathrm{d}t + (\nabla \operatorname{div} v, \nabla\tilde{w}) \\ &= \int_0^T [\partial_t \tilde{w}(t), \operatorname{div} v(t)]\, \mathrm{d}t + (\nabla\tilde{w}, \nabla \operatorname{div} v) = 0. \end{aligned}$$

Since $\varphi \in \mathrm{L}^{p'}(0,T;\widehat{W}^{1,q'}_{\omega'}(\Omega))$ was arbitrary, we deduce $\nabla\tilde{w} = \nabla w = 0$ and hence $w = 0$ by $w(0) = 0$.

3. Preliminaries

If equipped with addition as group operation and the canonical quotient topology inherited from $\mathbb{R}^n$, $G := \mathbb{R}^{n_1} \times \mathbb{T}_L^{n_2}$ is turned into a locally compact abelian group. Thus, under the canonical identification of G with $\mathbb{R}^{n_1} \times [0, L)^{n_2}$ the Haar measure μ on G is given up to a normalization factor by the product of the Lebesgue measure on $\mathbb{R}^{n_1}$ and the Lebesgue measure on $[0, L)^{n_2}$, that is

$$\int_G f \,\mathrm{d}\mu = \frac{1}{L^{n_2}} \int_{[0,L)^{n_2}} \int_{\mathbb{R}^{n-1}} f(x', x_n) \,\mathrm{d}x' \,\mathrm{d}y, \quad f \in C(G) \text{ with } \operatorname{supp} f \text{ compact.}$$

Let $\Omega \subset G$ be a domain, *i.e.*, an open connected subset of G. For $q \in [1, \infty]$ and a partially periodic Muckenhoupt weight $\omega \in A_q(G)$, the weighted Lebesgue space $\mathrm{L}^q_\omega(\Omega)$ is the space of all q-integrable functions with respect to the measure $\omega \,\mathrm{d}\mu$. Note here, that the classes $A_1(\mathbb{R}^n)$ and $A_\infty(\mathbb{R}^n)$ can be defined in a similar manner as for $q \in (1, \infty)$, see e.g. [24] for details on Muckenhoupt weights. The dual space of $\mathrm{L}^q_\omega(\Omega)$ can be identified with $\mathrm{L}^{q'}_{\omega'}(\Omega)$ via the duality pairing $(u, v) := \int_\Omega u\, v \,\mathrm{d}\mu$.

Since the topology of G is inherited by $\mathbb{R}^n$, we can talk in virtue of the canonical quotient mapping about the space of smooth functions $C^\infty(G)$ and the Schwartz–Bruhat space $\mathcal{S}(G)$ [4,19]. It is well-known that the Pontryagin dual of G is $\hat{G} = \mathbb{R}^{n_1} \times \Lambda_L^{n_2}$, where $\Lambda_L := \frac{2\pi}{L}\mathbb{Z}$. The differentiable structure on $\hat{G}$ and in particular the Schwartz–Bruhat space $\mathcal{S}(\hat{G})$ is introduced in a similar way as for G. We refer to [15,22] for details. We remark $\mathcal{S}(G) \hookrightarrow \mathrm{L}^q_\omega(G) \hookrightarrow \mathcal{S}'(G)$, see [22, Lemma 2] (and [21, Lemma 3.6] in the case $q = 1$).

We define weighted Sobolev spaces and homogeneous Sobolev spaces in terms of weak derivatives, that is

$$W^{m,q}_\omega(\Omega) := \{u \in \mathrm{L}^q_\omega(\Omega) \mid (\forall |\alpha| \le m)\ \partial^\alpha u \in \mathrm{L}^q_\omega(\Omega)\},$$
$$\|u\|_{W^{m,q}_\omega(\Omega)} := \sum_{|\alpha| \le m} \|\partial^\alpha u\|_{\mathrm{L}^q_\omega(\Omega)},$$

and

$$\widehat{W}^{m,q}_\omega(\Omega) := \{u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega) \mid (\forall |\alpha| = m)\ \partial^\alpha u \in \mathrm{L}^q_\omega(\Omega)\ \}/\!\sim,$$
$$\|u\|_{\widehat{W}^{m,q}_\omega(\Omega)} := \sum_{|\alpha| = m} \|\partial^\alpha u\|_{\mathrm{L}^q_\omega(\Omega)},$$

where the equivalence relation $\sim$ identifies two functions u_1 and u_2 whenever the norm of their difference vanishes. Moreover, we define the dual spaces $W^{-m,q}_{0,\omega}(\Omega) := [W^{m,q'}_{\omega'}(\Omega)]'$ and $\widehat{W}^{-m,q}_{0,\omega}(\Omega) := [\widehat{W}^{m,q'}_{\omega'}(\Omega)]'$, equipped with the corresponding dual norms. The duality pairing we denote by $[u, \varphi]$.

Remark 3.1. It should be noted that for all $q \in [1, \infty]$ and $\omega \in A_q(G)$, $W^{0,q}_\omega(\Omega) = L^q_\omega(\Omega)$, and that for all $m \in \mathbb{N}_0$ the spaces $W^{m,q}_\omega(\Omega)$ and $\widehat{W}^{m,q}_\omega(\Omega)$ equipped with their respective norms yield Banach spaces. Moreover, Lemma 3 in [22] shows that $C_0^\infty(G)$ is dense in $W^{m,q}_\omega(G)$ as long as $q \in (1, \infty)$. Since the approximating sequence constructed there depends neither on the exponent of integrability nor on the Muckenhoupt weight, we see that $C_0^\infty(G)$ is even dense in $W^{m,q_1}_{\omega_1}(G) \cap W^{m,q_2}_{\omega_2}(G)$ for $q_i \in (1, \infty)$ and $\omega_i \in A_{q_i}(G)$, respectively.

Muckenhoupt weights behave well under mirroring: For a generic function φ on G, let us define $\varphi^*(x) := \varphi(-x_1, x_2, \ldots, x_n)$, $x \in G$.

Lemma 3.2. *Let $q \in (1, \infty)$ and $\omega \in A_q(\mathbb{R}^n)$ and define*

$$\widehat{\omega} := \begin{cases} \omega(x), & \text{if } x_1 \ge 0, \\ \omega(-x_1, x_2, \ldots, x_n), & \text{if } x_1 < 0. \end{cases}$$

Then $\widehat{\omega} \in A_q(\mathbb{R}^n)$ and we have the estimate $\mathcal{A}_q(\widehat{\omega}) \le 2^q \mathcal{A}_q(\omega)$. Moreover $\widehat{\omega} = \widehat{\omega}^*$.

Proof. See [10, Lemma 2.1]. □

By the canonical identification of G and $\mathbb{R}^{n_1} \times [0, L)^{n_2}$, we can associate to any domain $\Omega \subset G$ a domain $\tilde{\Omega} \subset \mathbb{R}^n$. It is instructive to think of $\tilde{\Omega}$ as one periodic cell of the domain Ω. We call a subset $\Omega \subset G$ a (bounded) Lipschitz domain, if the corresponding $\tilde{\Omega} \subset \mathbb{R}^n$ is a (bounded) Lipschitz domain.

Moreover, we divide the boundary of $\tilde{\Omega}$ into the two parts Σ and $\partial\tilde{\Omega}_G$, where Σ are the faces at the end of the cell (if there are such) and $\partial\tilde{\Omega}_G$ coincides with $\partial\Omega$ under the canonical identification of G and $\mathbb{R}^{n_1} \times [0, L)^{n_2}$.

For bounded domains, weighted spaces can be embedded into non-weighted ones by the open-ended property of Muckenhoupt weights.

Lemma 3.3. *Let $\Omega \subset G$ be a bounded open set, $q \in (1, \infty)$ and let $\omega \in A_q(G)$. Then there is $1 < r < \infty$ such that $\mathrm{L}^r(\Omega) \hookrightarrow \mathrm{L}^q_\omega(\Omega)$.*

Furthermore, there exists $\varepsilon_0 > 0$ such that $\mathrm{L}^q_\omega(\Omega) \hookrightarrow L^{1+\varepsilon}(\Omega)$ for all $0 \le \varepsilon \le \varepsilon_0$. Here, $1/\varepsilon_0 > 0$ is A_q-consistent.

Moreover, for all $\omega \in A_q(G)$ with $\mathcal{A}_q(\omega) \le C < \infty$ and $\omega(Q) \ge c > 0$, where Q denotes a cube with $\tilde{\Omega} \subset Q$, the embedding constant of the embedding $\mathrm{L}^q_\omega(\Omega) \hookrightarrow L^{1+\varepsilon}(\Omega)$ can be chosen uniformly.

Proof. In view of the canonical identification of Ω and $\tilde{\Omega}$, the result follows immediately from the corresponding non-periodic result in [11, Lemma 2.2] on $\tilde{\Omega}$. □

Let $\Omega \subset G$ be a (possibly unbounded) Lipschitz domain. We use Lemma 3.3 to introduce the function spaces $W^{1,q}_{0,\omega}(\Omega)$ and $\widehat{W}^{1,q}_{0,\omega}(\Omega)$ in the canonical way, namely as the subspaces of $W^{1,q}_\omega(\Omega)$ (resp. $\widehat{W}^{1,q}_\omega(\Omega)$) of functions whose trace vanishes locally. Again, we introduce the corresponding dual spaces $W^{-1,q}_\omega(\Omega) := [W^{1,q'}_{0,\omega'}(\Omega)]'$ and $\widehat{W}^{-1,q}_\omega(\Omega) := [\widehat{W}^{1,q'}_{0,\omega'}(\Omega)]'$ with corresponding dual norms.

Lemma 3.4. *Let $\Omega \subset G$ be a bounded Lipschitz domain, $q \in (1, \infty)$, $\omega \in A_q(G)$, and let $h : W^{1,1}(\tilde{\Omega}) \to [0, \infty]$ be a continuous semi-norm such that $h(c) = 0$ implies $c = 0$ for constant functions c. Then there is an A_q-consistent $C(n, q, \omega, \Omega) > 0$ such that*

$$\|u\|_{\mathrm{L}^q_\omega(\Omega)} \le C\|\nabla u\|_{\mathrm{L}^q_\omega(\Omega)}$$

for all $u \in W^{1,q}_\omega(\Omega)$ with $h(u) = 0$.

Proof. See [11, Corollary 2.1] for the corresponding non-periodic result. □

Corollary 3.5. *Let $\Omega \subset G$ be a bounded Lipschitz domain, $q \in (1, \infty)$ and $\omega \in A_q(G)$. Then there is an A_q-consistent $C(n, q, \omega, \Omega) > 0$ such that for all $v \in W^{1,q}_\omega(\Omega)$ it holds $\|v\|_{\mathrm{L}^q_\omega(\Omega)} \le C\|\nabla v\|_{\mathrm{L}^q_\omega(\Omega)}$ if*

(i) $\int_\Omega v \, d\mu = 0$ *or*

(ii) $v \in W^{1,q}_\omega(\Omega) \cap \widehat{W}^{-1,q}_{0,\omega}(\Omega)$ *or*

(iii) $v \in W^{1,q}_{0,\omega}(\Omega)$.

Proof. Follows by Lemma 3.4. For part (ii) recall that if $v \in W^{1,q}_\omega(\Omega) \cap \widehat{W}^{-1,q}_{0,\omega}(\Omega)$ then $\int_\Omega v \, \mathrm{d}\mu = 0$, since otherwise $[v, \mathbb{1}] = \int_\Omega v \, \mathrm{d}\mu \neq 0$, but $\|\nabla \mathbb{1}\|_{\mathrm{L}^{q'}_{\omega'}(\Omega)} = 0$, which is a contradiction. □

Lemma 3.6. *Let $\Omega \subset G$ be a bounded Lipschitz domain, $q \in (1, \infty)$ and $\omega \in A_q(G)$. Then there is an A_q-consistent $C(n, q, \omega, \Omega) > 0$ such that for all $u \in W^{2,q}_\omega(\Omega) \cap W^{1,p}_{0,\omega}(\Omega)$ it holds $\|u\|_{W^{2,q}_\omega(\Omega)} \le C\|\nabla^2 u\|_{\mathrm{L}^q_\omega(\Omega)}$.*

Proof. The same assertion has been proven in [11, Corollary 2.2] in the non-periodic setting for $u \in W^{2,q}_\omega(\tilde{\Omega})$ with $u|_{\partial\tilde{\Omega}} = 0$, *i.e.*, u vanishes on the whole of $\partial\tilde{\Omega}$. Revising the proof, we see that it suffices that u vanishes on $\partial\tilde{\Omega}_G$. □

Lemma 3.7. *Let $\tilde{\Omega} \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $q \in (1, \infty)$ and $\{\omega_j\}_{j\in\mathbb{N}} \subset A_q(\mathbb{R}^n)$ such that*

$$\sup_{j\in\mathbb{N}} \mathcal{A}_q(\omega_j) < \infty \quad \text{and} \quad (\forall m \in \mathbb{N}) \ \omega_j(Q) = 1,$$

where Q is an open cube with $\overline{\tilde{\Omega}} \subset Q$. If $\{u_j\}_{j\in\mathbb{N}} \subset W^{1,q}_{\omega_j}(\tilde{\Omega})$ is bounded, and we have the weak convergence $u_j \rightharpoonup 0$ in $W^{1,s}(\tilde{\Omega})$ for some $1 < s < \infty$, then $\|u_j\|_{L^q_{\omega_j}(\tilde{\Omega})} \to 0$.

Proof. See [11, Theorem 2.4]. □

4. The Whole Space

Recall the partially periodic weighted Mikhlin theorem from [22].

Theorem 4.1. *Suppose that $M \in C^n(\mathbb{R}^n \setminus \{0\})$ is such that the origin 0 belongs to the Lebesgue set of M, and that there is a constant $c > 0$ such that for all multi-indices α with $|\alpha| \leq n$ and all $\xi \in \mathbb{R}^n \setminus \{0\}$ it holds $|\xi|^{|\alpha|}|D^\alpha M(\xi)| < c$. Then for every $q \in (1,\infty)$ and $\omega \in A_q(\mathbb{R}^n)$, $m := M|_{\hat{G}}$ is an $\mathrm{L}^q_\omega(G)$-multiplier with an A_q-consistent bound.*

Proof. This is the nonperiodic weighted Mikhlin theorem [14, Theorem 2], [12, Chapter IV, Theorem 3.9] combined with the transference principle in [22, Proposition 4, Remark 5]. □

Lemma 4.2. *Let $q \in (1,\infty)$ and $\omega \in A_q(G)$. Then there is an A_q-consistent constant $c = c(n, q, \omega) > 0$ such that for all $u \in \widehat{W}^{2,q}_\omega(G) \cap \mathrm{L}^q_\omega(G)$ and all $\varepsilon > 0$ it holds*

$$\|\nabla u\|_{\mathrm{L}^q_\omega(G)} \leq c\left(\frac{1}{\varepsilon}\|u\|_{\mathrm{L}^q_\omega(G)} + \varepsilon\|\nabla^2 u\|_{\mathrm{L}^q_\omega(G)}\right).$$

In particular, $W^{2,q}_\omega(G) = \widehat{W}^{2,q}_\omega(G) \cap \mathrm{L}^q_\omega(G)$.

Proof. Defining $M : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{C}$ via

$$M(\zeta) := \frac{i\zeta}{\frac{1}{\varepsilon} + \varepsilon|\zeta|^2},$$

we see that $\nabla u = \mathcal{F}^{-1}_G[M|_{\hat{G}} \cdot \mathcal{F}_G[\frac{1}{\varepsilon}u - \varepsilon\Delta u]]$. Since M fulfills the Mikhlin condition with a bound independent of ε (which is readily seen from $|\zeta| \leq \frac{1}{\varepsilon} + \varepsilon|\zeta|^2$ and $\varepsilon|\zeta|^2 \leq \frac{1}{\varepsilon} + \varepsilon|\zeta|^2$), the assertion follows from Theorem 4.1. □

Multipliers that are smooth only outside the origin play an important rôle in the field of partial differential equations. Therefore, we state the following theorem on 0-homogeneous multipliers.

Theorem 4.3. *Let $M \in C^n(\mathbb{R}^n \setminus \{0\})$ be homogeneous of degree 0. Then the origin $0 \in \mathbb{R}^n$ is contained in the Lebesgue set of M. In particular, for $q \in (1,\infty)$ and $\omega \in A_q(G)$ the partially periodic Riesz transformations R_j, $j \in \{1,\ldots,n\}$, defined via $R_j = \mathcal{F}^{-1}_G m_j \mathcal{F}_G$ with*

$$m_j : \hat{G} \to \mathbb{C}, \quad m_j(\eta) := \begin{cases} 0, & \text{if } \eta = 0, \\ \mathrm{i}\frac{\eta_j}{|\eta|}, & \text{else,} \end{cases}$$

extend to bounded operators on $\mathrm{L}^q_\omega(G)$ with an A_q-consistent bound.

Proof. Due to the homogeneity of M it holds

$$\lim_{r\to 0}\frac{1}{|B_r(0)|}\int_{B_r(0)} M(\zeta)\,\mathrm{d}\zeta = \frac{1}{|B_1(0)|}\int_{B_1(0)} M(\zeta)\,\mathrm{d}\zeta,$$

which shows that 0 is in the Lebesgue set of M.

For the assertion about the Riesz transformation, define

$$M_j : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}, \quad M_j(\zeta) := \mathrm{i}\frac{\zeta_j}{|\zeta|}.$$

We note that $m_j = M_j|_{\hat{G}}$, where for $\eta = 0$ this is to be understood in the sense that $\lim_{r\to 0}\frac{1}{|B_r(0)|}\int_{B_r(0)} M_j(\zeta)\,\mathrm{d}\zeta = 0 = m_j(0)$. Therefore, Theorem 4.1 yields the assertion. □

Corollary 4.4. *Let $n \geq 2$, $q \in (1,\infty)$ and $\omega \in A_q(G)$. Then there is an A_q-consistent $c = c(n,q,\omega) > 0$ such that for all $u \in \widehat{W}^{2,q}_{\omega}(G)$ it holds*

$$\|\nabla^2 u\|_{\mathrm{L}^q_\omega(G)} \leq c\|\Delta u\|_{\mathrm{L}^q_\omega(G)}. \tag{4.1}$$

Proof. Since $\partial_i\partial_j u = R_iR_j(\Delta u)$, the assertion follows from Theorem 4.3. □

Lemma 4.5. *Let $q_i \in (1,\infty)$ and $\omega_i \in A_{q_i}(G)$, respectively, where $i = 1,2$.*

(i) *If $u \in \mathrm{L}^{q_1}_{\omega_1}(G) + \mathrm{L}^{q_2}_{\omega_2}(G)$ is harmonic, then $u = 0$.*

(ii) *Let $u \in \mathrm{L}^1_{\mathrm{loc}}(G)$ with $\int_G u\Delta\varphi\, d\mu = 0$ for all $\varphi \in C_0^\infty(G)$. Then u is harmonic. In particular, the space $\Delta C_0^\infty(G)$ is dense in $\mathrm{L}^{q_1}_{\omega_1}(G) \cap \mathrm{L}^{q_2}_{\omega_2}(G)$.*

Proof. (i) Let us first assume $q_1 = q_2 =: q$ and $\omega_1 = \omega_2 =: \omega$. Since $\omega' \in A_{\frac{q'}{1+\varepsilon}}(G)$ for some $\varepsilon > 0$ by the open-ended property of Muckenhoupt weights, we obtain with $P(x) := (L + |x'|)^{n_1}$ as in the proof of Lemma 2 in [22] the estimate

$$\int_G P^{-\frac{q'}{1+\varepsilon}}\omega'\,\mathrm{d}\mu < \infty.$$

Next, for $k \in \mathbb{N}$, the volume of a cuboid U_k with edges of length 2^k in the direction of the variables x' and length L in the direction of y, can be computed as $\mu(U_k) = 2^{kn_1}$. Also, the function P is bounded on U_k by $\|P\|_{\mathrm{L}^\infty(U_k)} \lesssim 2^{kn_1}$, where $\lesssim$ means that it can be estimated modulo a constant $c = c(n_1, L)$. Since $u \in \mathrm{L}^q_\omega(G)$ is harmonic, we obtain by the mean value formula

$$\begin{aligned}|u(0)| &\lesssim \frac{1}{\mu(U_k)}\int_{U_k}|u|\,\mathrm{d}\mu \lesssim 2^{-kn_1}\|P^{\frac{1}{1+\varepsilon}}\|_{\mathrm{L}^\infty(U_k)}\|u\|_{\mathrm{L}^q_\omega(G)}\\ &\lesssim 2^{-kn_1\frac{\varepsilon}{1+\varepsilon}}\|u\|_{\mathrm{L}^q_\omega(G)}.\end{aligned}$$

Sending $k \to \infty$ yields $u(0) = 0$. Similarly, we obtain $u(x) = 0$ for all $x \in G$. If $u = u_1 + u_2 \in \mathrm{L}^{q_1}_{\omega_1}(G) + \mathrm{L}^{q_2}_{\omega_2}(G)$, then we use the same computation as above for

$$|u(0)| \leq \frac{c}{\mu(U_k)}\int_{U_k}|u_1|\,\mathrm{d}\mu + \frac{c}{\mu(U_k)}\int_{U_k}|u_2|\,\mathrm{d}\mu.$$

(ii) Let $u \in \mathrm{L}^1_{\mathrm{loc}}(G)$ satisfy $\int_G u\Delta\varphi\,\mathrm{d}\mu = 0$ for all $\varphi \in C_0^\infty(G)$. Let $\psi \in C_0^\infty(B_\rho)$, where $B_\rho \subset \mathbb{R}^n$ is a ball of small radius $\rho \ll L$. Then ψ can be extended to a periodic function, and hence $\int_{B_\rho} u\Delta\psi\,\mathrm{d}x = 0$. Therefore, by Weyl's Lemma, u is harmonic in B_ρ. Since the origin of the ball was arbitrary, u is harmonic everywhere.

In order to show the density of $\Delta C_0^\infty(G)$ in $\mathrm{L}^{q_1}_{\omega_1}(G) \cap \mathrm{L}^{q_2}_{\omega_2}(G)$, consider a function $v \in \mathrm{L}^{q_1'}_{\omega_1'}(G) + \mathrm{L}^{q_2'}_{\omega_2'}(G)$ with $\int_G v\Delta\varphi\,\mathrm{d}\mu = 0$ for all $\varphi \in C_0^\infty(G)$. Then v is harmonic and by part (i) it follows $v = 0$. Hahn–Banach's theorem yields the assertion.

□

Remark 4.6. Since Weyl's Lemma is true for arbitrary open subsets of $\mathbb{R}^n$, a completely analogous argument to the one given in the proof of Lemma 4.5(ii) shows that for any open subset $\Omega \subset G$ it holds that $u \in \mathrm{L}^1_{\mathrm{loc}}(G)$ is harmonic in Ω, if $\int_\Omega u\Delta\varphi\,\mathrm{d}\mu = 0$ for all $\varphi \in C_0^\infty(\Omega)$.

4.1. Weak Solutions to the Laplace Equation

Consider the weak Laplace operator

$$\begin{aligned}&\Delta_{q,\omega} : \widehat{W}^{1,q}_\omega(G) \to \widehat{W}^{-1,q}_\omega(G)\\ &(\Delta_{q,\omega}u)(\varphi) := -(\nabla u, \nabla\varphi), \quad \varphi \in \widehat{W}^{1,q'}_{\omega'}(G),\end{aligned}$$

where $q \in (1,\infty)$ and $\omega \in A_q(G)$.

Proposition 4.7. *Let $q, q_i \in (1,\infty)$ and $\omega \in A_q(G)$, $\omega_i \in A_{q_i}(G)$, respectively, where $i = 1, 2$.*

(i) *The operator $\Delta_{q,\omega} : \widehat{W}^{1,q}_{\omega}(G) \to \widehat{W}^{-1,q}_{\omega}(G)$ is an isomorphism and there is an A_q-consistent $c = c(n, q, \omega) > 0$ such that*

$$\|\nabla u\|_{\mathrm{L}^q_\omega(G)} \le c\|\Delta_{q,\omega} u\|_{\widehat{W}^{-1,q}_\omega(G)}, \tag{4.2}$$

for all $u \in \widehat{W}^{1,q}_\omega(G)$.

Moreover, the adjoint operator $\Delta'_{q,\omega} : \widehat{W}^{1,q'}_{\omega'}(G) \to \widehat{W}^{-1,q'}_{\omega'}(G)$ coincides with $\Delta_{q',\omega'}$.

(ii) *If $F \in \widehat{W}^{-1,q_1}_{\omega_1}(G) \cap \widehat{W}^{-1,q_2}_{\omega_2}(G)$, then the weak solution of $\Delta u = -F$ satisfies $u \in \widehat{W}^{1,q_1}_{\omega_1}(G) \cap \widehat{W}^{1,q_2}_{\omega_2}(G)$.*

Proof. (i) Clearly, $\Delta_{q,\omega}$ is a bounded operator. Since $\Delta C_0^\infty(G)$ is dense in $\mathrm{L}^{q'}_{\omega'}(G)$ by Lemma 4.5, we obtain

$$\begin{aligned}\|\partial_j u\|_{\mathrm{L}^q_\omega(G)} &= \sup_{0\neq\varphi\in C_0^\infty(G)} \frac{|(\partial_j u, \Delta\varphi)|}{\|\Delta\varphi\|_{\mathrm{L}^{q'}_{\omega'}(G)}}\\ &\le c \sup_{0\neq\varphi\in C_0^\infty(G)} \frac{|[-\Delta_{q,\omega}u, \partial_j\varphi]|}{\|\nabla\partial_j\varphi\|_{\mathrm{L}^{q'}_{\omega'}(G)}} \le c\|\Delta_{q,\omega}u\|_{\widehat{W}^{-1,q}_\omega(G)},\end{aligned}$$

for $j \in \{1,\dots,n\}$, where we have used Corollary 4.4, which also gives $A_{q'}(G)$-consistency and hence $A_q(G)$-consistency of the constant c. This shows

$$\|\nabla u\|_{\mathrm{L}^q_\omega(G)} \le c\|\Delta_{q,\omega}u\|_{\widehat{W}^{-1,q}_\omega(G)}.$$

Therefore, $\Delta_{q,\omega}$ is injective and has closed range. By reasons of symmetry it holds $\Delta'_{q,\omega} = \Delta_{q',\omega'}$ and thus also the adjoint operator is injective and has closed range. Consequently, $\Delta_{q,\omega}$ is an isomorphism by the closed range theorem.

(ii) Let $u_i \in \widehat{W}^{q_i}_{\omega_i}(G)$, $i = 1, 2$, denote the corresponding solutions of $\Delta u_i = -F$. Then $(u_1 - u_2, \Delta\varphi) = 0$ for all $\varphi \in C_0^\infty(G)$, and hence $u_1 - u_2$ is harmonic by Lemma 4.5(ii). Then also the gradient $\nabla(u_1 - u_2) \in \mathrm{L}^{q_1}_{\omega_1}(G) + \mathrm{L}^{q_2}_{\omega_2}(G)$ is harmonic and thus $\nabla u_1 = \nabla u_2$ by Lemma 4.5(i).

$\square$

Corollary 4.8. *Let $q_i \in (1,\infty)$ and $\omega_i \in A_{q_i}(G)$, respectively, where $i = 1, 2$.*

(i) *$C_0^\infty(G)$ is dense in $\widehat{W}^{1,q_1}_{\omega_1}(G) \cap \widehat{W}^{1,q_2}_{\omega_2}(G)$.*

(ii) *$C_0^\infty(G)$ is dense in $\widehat{W}^{2,q_1}_{\omega_1}(G) \cap \widehat{W}^{2,q_2}_{\omega_2}(G)$.*

Proof. (i) Let

$$F = F_1 + F_2 \in \widehat{W}^{-1,q'_1}_{\omega'_1}(G) + \widehat{W}^{-1,q'_2}_{\omega'_2}(G) = (\widehat{W}^{1,q_1}_{\omega_1}(G) \cap \widehat{W}^{1,q_2}_{\omega_2}(G))'$$

be such that $[F, \varphi] = 0$ for all $\varphi \in C_0^\infty(G)$. By Proposition 4.7 we may define $u_i = \Delta^{-1}_{q,\omega}F_i \in \widehat{W}^{1,q'_i}_{\omega'_i}(G)$, respectively. Lemma 4.5(ii) shows that $u_1 + u_2$ and consequently also $\nabla(u_1 + u_2) \in L^{q'_1}_{\omega'_1}(G) + L^{q'_1}_{\omega'_1}(G)$ is harmonic. Thus, $u_1 + u_2 = 0$ in $\widehat{W}^{1,q'_1}_{\omega'_1}(G) \cap \widehat{W}^{1,q'_2}_{\omega'_2}(G)$ and so $F = 0$. An application of Hahn–Banach's theorem yields the assertion.

(ii) Let $u \in \widehat{W}^{2,q_1}_{\omega_1}(G) \cap \widehat{W}^{2,q_2}_{\omega_2}(G)$. By Lemma 4.5(ii) there is a corresponding sequence $\{\varphi_k\}_{k\in\mathbb{N}} \subset C_0^\infty(G)$ such that $\Delta\varphi_k \to \Delta u$ in $\mathrm{L}^{q_1}_{\omega_1}(G) \cap \mathrm{L}^{q_2}_{\omega_2}(G)$ as $k \to \infty$. Thus, Corollary 4.4 implies, that $\varphi_k \to u$ in $\widehat{W}^{2,q}_\omega(G)$ as $k \to \infty$.

$\square$

Let us now turn to the resolvent problem of the Laplace equation. Assume $\lambda \in \Sigma_\vartheta$ for some $\vartheta \in (0, \pi)$ and consider the operator

$$\begin{aligned}&(\lambda - \Delta)_{q,\omega} : W^{1,q}_\omega(G) \to W^{-1,q}_\omega(G)\\ &\quad[(\lambda - \Delta)_{q,\omega}u, \varphi] := \lambda(u, \varphi) + (\nabla u, \nabla\varphi), \quad \varphi \in W^{1,q'}_{\omega'}(G).\end{aligned}$$

Proposition 4.9. *Let $q, q_i \in (1,\infty)$, $\omega \in A_q(G)$, $\omega_i \in A_{q_i}(G)$, $i = 1, 2$, respectively, $\vartheta \in (0,\pi)$ and $\lambda \in \Sigma_\vartheta$.*

(i) *$(\lambda - \Delta)_{q,\omega}$ is an isomorphism. Moreover, there is an A_q-consistent constant $c = c(n, q, \omega, \vartheta) > 0$ such that*

$$\min\{|\lambda|, \sqrt{|\lambda|}\}\|u\|_{\mathrm{L}^q_\omega(G)} + \min\{\sqrt{|\lambda|}, 1\}\|\nabla u\|_{\mathrm{L}^q_\omega(G)} \le c\|(\lambda - \Delta)_{q,\omega} u\|_{W^{-1,q}_\omega(G)}.$$

(ii) *If $F \in W^{-1,q_1}_{\omega_2}(G) \cap W^{-1,q_2}_{\omega_2}(G)$, then the weak solution of $\lambda u - \Delta u = F$ satisfies $u \in W^{1,q_1}_{\omega_1}(G) \cap W^{1,q_2}_{\omega_2}(G)$.*

(iii) *Viewed as an operator from $W^{1,q}_\omega(G) \cap \widehat{W}^{-1,q}_\omega(G)$ to $\widehat{W}^{-1,q}_\omega(G)$, $(\lambda - \Delta)_{q,\omega}$ is still an isomorphism and there is an A_q-consistent constant $c = c(n, q, \omega, \vartheta) > 0$ such that*

$$|\lambda|\|u\|_{\widehat{W}^{-1,q}_\omega(G)} + \sqrt{|\lambda|}\|u\|_{\mathrm{L}^q_\omega(G)} + \|\nabla u\|_{\mathrm{L}^q_\omega(G)} \le c\|(\lambda - \Delta)_{q,\omega} u\|_{\widehat{W}^{-1,q}_\omega(G)}.$$

Proof. (i) If $(\lambda - \Delta)u = 0$ for $u \in \mathcal{S}'(G)$, then an application of the Fourier transform gives $u = 0$, which shows the injectivity of $(\lambda - \Delta)_{q,\omega}$.

Concerning the surjectivity, we find for $F \in W^{-1,q}_\omega(G)$ functions $f_0, f_1, \ldots, f_n \in \mathrm{L}^q_\omega(G)$ such that

$$[F, \varphi] = (f_0, \varphi) + \sum_{i=0}^{n} (f_i, \partial_i \varphi), \quad \varphi \in W^{1,q'}_{\omega'}(G),$$

$$\sum_{i=0}^{n} \|f_i\|_{\mathrm{L}^q_\omega(G)} \le C\|F\|_{W^{-1,q}_\omega(G)},$$

where $C = C(n) > 0$ is independent of ω. By [22, Theorem 1], there are $u_i \in W^{2,q}_\omega(G)$ such that $(\lambda - \Delta)u_i = f_i$ for $i \in \{0, \ldots, n\}$ with corresponding estimates

$$\|\lambda u_i, \sqrt{|\lambda|}\nabla u_i, \nabla^2 u_i\|_{\mathrm{L}^q_\omega(G)} \le c\|f_i\|_{\mathrm{L}^q_\omega(G)}. \tag{4.3}$$

Here, the constant $c = c(n, q, \omega, \vartheta) > 0$ is A_q-consistent. We conclude that for $u := u_0 - \sum_{i=1}^n \partial_i u_i \in W^{1,q}_\omega(G)$ it holds

$$[F, \varphi] = \lambda(u, \varphi) + (\nabla u, \nabla \varphi), \quad \varphi \in C_0^\infty(G).$$

By density, this extends to all $\varphi \in W^{1,q'}_{\omega'}(G)$. Furthermore, estimates (4.3) give

$$\min\{|\lambda|, \sqrt{|\lambda|}\}\|u\|_{\mathrm{L}^q_\omega(G)} \le c\left(\sum_{i=0}^{n} \|f_i\|_{\mathrm{L}^q_\omega(G)}\right) \le c\|F\|_{W^{-1,q}_\omega(G)},$$

$$\min\{\sqrt{|\lambda|}, 1\}\|\nabla u\|_{\mathrm{L}^q_\omega(G)} \le c\left(\sum_{i=0}^{n} \|f_i\|_{\mathrm{L}^q_\omega(G)}\right) \le c\|F\|_{W^{-1,q}_\omega(G)}.$$

(ii) Let $u_i := (\lambda - \Delta)^{-1}_{q_i,\omega_i} F \in W^{1,q_i}_{\omega_i}(G)$, $i = 1, 2$. Then the difference $v := u_1 - u_2 \in \mathcal{S}'(G)$ satisfies $(\lambda - \Delta)v = 0$ in the sense of tempered distributions. An application of the Fourier transform shows that $v = 0$ and hence $u_1 = u_2$.

(iii) The proof follows analogously as in part (i), only without f_0 and u_0. Then $\sqrt{|\lambda|}\|u\|_{\mathrm{L}^q_\omega(G)}$ and $\|\nabla u\|_{\mathrm{L}^q_\omega(G)}$ can be estimated by $\|F\|_{\widehat{W}^{-1,q}_\omega(G)}$ as before, while the estimate for $|\lambda|\|u\|_{\widehat{W}^{-1,q}_\omega(G)}$ follows from $\lambda u = F - (\nabla u, \nabla \cdot)$.

□

Let us conclude this section with a regularity result. For a functional $F \in \widehat{W}^{-1,q}_\omega(G)$ and $j \in \{1, \ldots, n\}$ we define $\partial_j F \in \mathcal{S}'(G)$ via

$$[\partial_j F, \varphi] := -[F, \partial_j \varphi], \quad \varphi \in \mathcal{S}(G).$$

Corollary 4.10. *Let $j \in \{1, \ldots, n\}$, $q \in (1,\infty)$, $\omega \in A_q(G)$, $\vartheta \in (0,\pi)$ and $\lambda \in \Sigma_\vartheta$. Moreover assume $\hat{F} \in \widehat{W}^{-1,q}_\omega(G)$ and $F \in W^{-1,q}_\omega(G)$. Then $u := \Delta^{-1}_{q,\omega}\hat{F} \in \widehat{W}^{1,q}_\omega(G)$ and $u_\lambda := (\lambda - \Delta)^{-1}_{q,\omega} F \in W^{1,q}_\omega(G)$ are well-defined by Propositions 4.7 and 4.9.*

(i) *Assume that additionally* $\partial_j \hat{F} \in \widehat{W}_\omega^{-1,q}(G)$*. Then* $\partial_j u \in W_\omega^{1,q}(G)$ *and we have* $\partial_j u = \Delta_{q,\omega}^{-1} \partial_j \hat{F}$*. Moreover, there is an* A_q*-consistent* $c = c(n, q, \omega) > 0$ *such that*

$$\|\nabla \partial_j u\|_{\mathrm{L}^q_\omega(G)} \leq c \|\partial_j \hat{F}\|_{\widehat{W}_\omega^{-1,q}(G)}.$$

(ii) *Assume that additionally* $\partial_j F \in W_\omega^{-1,q}(G)$*. Then* $\partial_j u_\lambda \in W_\omega^{1,q}(G)$ *and we have* $\partial_j u_\lambda = (\lambda - \Delta)_{q,\omega}^{-1} \partial_j F$*. Moreover, there is an* A_q*-consistent* $c = c(n, q, \omega, \vartheta) > 0$ *such that*

$$\min\{|\lambda|, \sqrt{|\lambda|}\} \|\partial_j u_\lambda\|_{\mathrm{L}^q_\omega(G)} + \min\{\sqrt{|\lambda|}, 1\} \|\nabla \partial_j u_\lambda\|_{\mathrm{L}^q_\omega(G)} \leq c \|\partial_j F\|_{W_\omega^{-1,q}(G)}.$$

Proof. (i) Denote by $v \in W_\omega^{1,q}(G)$ the unique solution to

$$(v, \varphi) + (\nabla v, \nabla \varphi) = (\partial_j u, \varphi) - [\partial_j \hat{F}, \varphi], \quad \varphi \in C_0^\infty(G), \tag{4.4}$$

which is well-defined by Proposition 4.9 (with $\lambda = 1$) and due to the facts $\partial_j F \in \widehat{W}_\omega^{-1,q}(G) \subset W_\omega^{-1,q}(G)$ and $\partial_j u \in \mathrm{L}^q_\omega(G) \subset W_\omega^{-1,q}(G)$. In particular, it holds

$$\begin{aligned}(v, \varphi) + (\nabla v, \nabla \varphi) &= (\partial_j u, \varphi) - [\partial_j \hat{F}, \varphi] = (\partial_j u, \varphi) + [\hat{F}, \partial_j \varphi] \\ &= (\partial_j u, \varphi) - (\nabla u, \nabla \partial_j \varphi) = (\partial_j u, \varphi) + [\nabla \partial_j u, \nabla \varphi],\end{aligned}$$

and so $(1 - \Delta)(v - \partial_j u) = 0$ as an identity in $\mathcal{S}'(G)$. Hence, applying the Fourier transform, we see that $\partial_j u = v \in W_\omega^{1,q}(G)$, which proves the first claim. Relation (4.4) yields

$$(\nabla v, \nabla \varphi) = -[\partial_j \hat{F}, \varphi], \quad \varphi \in C_0^\infty(G),$$

and thus Proposition 4.7 shows $-\Delta_{q,\omega}^{-1} \partial_j \hat{F} = v = \partial_j u \in W_\omega^{1,q}(G)$ and

$$\|\nabla \partial_j u\|_{\mathrm{L}^q_\omega(G)} = \|\nabla v\|_{\mathrm{L}^q_\omega(G)} \leq c \|\partial_j \hat{F}\|_{\widehat{W}_\omega^{-1,q}(G)},$$

where $c = c(n, q, \omega) > 0$ is A_q-consistent.

(ii) Analogous.

□

Remark 4.11. Let $j \in \{1, \ldots, n\}$, $q \in (1, \infty)$ and $\omega \in A_q(G)$. A sufficient condition for $F \in \widehat{W}_\omega^{-1,q}(G)$ to satisfy $\partial_j F \in \widehat{W}_\omega^{-1,q}(G)$ is $F \in \mathrm{L}^q_\omega(G)$. Moreover, for all $F \in \mathrm{L}^q_\omega(G)$ both $F \in W_\omega^{-1,q}(G)$ and $\partial_j F \in W_\omega^{-1,q}(G)$ hold.

Corollary 4.12. *Let* $q, q_i \in (1, \infty)$ *and* $\omega \in A_q(G)$, $\omega_i \in A_{q_i}(G)$*, respectively, where* $i = 1, 2$.

(i) *If* $f \in \mathrm{L}^q(G)$ *has compact support and* $\int_G f \, d\mu = 0$*, then it holds* $f \in \widehat{W}^{-1,q}(G)$.

(ii) *The space* $\mathrm{L}^q(G) \cap \widehat{W}^{-1,q}(G)$ *is dense in* $\mathrm{L}^q(G)$.

(iii) *For every* $f \in \mathrm{L}^q_\omega(G)$ *there is a unique* $u \in \widehat{W}_\omega^{2,q}(G)$ *such that* $-\Delta u = f$*. Moreover, there is an* A_q*-consistent* $c = c(n, q, \omega) > 0$ *such that*

$$\|\nabla^2 u\|_{\mathrm{L}^q_\omega(G)} \leq c \|f\|_{\mathrm{L}^q_\omega(G)}.$$

(iv) *If* $f \in \mathrm{L}^{q_1}_{\omega_1}(G) \cap \mathrm{L}^{q_2}_{\omega_2}(G)$*, then the unique solution* $u \in \widehat{W}_{\omega_1}^{2,q_1}(G)$ *to* $-\Delta u = f$ *satisfies* $u \in \widehat{W}_{\omega_2}^{2,q_2}(G)$.

Proof. (i) Let $U \subset G$ be a smooth, bounded and such that $\operatorname{supp} f \subset U$. For $v \in \widehat{W}^{1,q'}(G) \subset \mathrm{L}^1_{\mathrm{loc}}(G)$ set $v_U := \int_U v \, \mathrm{d}\mu$. By Poincaré's inequality it follows

$$|[f, v]| = \left| \int_G f v \, \mathrm{d}\mu \right| = \left| \int_U f(v - v_U) \, \mathrm{d}\mu \right| \leq c \|f\|_{\mathrm{L}^q(G)} \|\nabla v\|_{\mathrm{L}^{q'}(G)},$$

which shows $f \in \widehat{W}^{-1,q}(G)$.

(ii) By truncation, a function in $\mathrm{L}^q(G)$ can be approximated by functions in $\mathrm{L}^q(G)$ with compact support. Hence, let $f \in \mathrm{L}^q(G)$ have compact support and set $(f) := \int_G f \,\mathrm{d}\mu$. For $R > 0$, let $Q_R \subset G$ be a cuboid with length R^{1/n_1} in direction of the variables x' and length L in direction of the variables y and consider the characteristic function χ_{Q_R}. If we write

$$f_R := \frac{(f)}{R}\chi_{Q_R},$$

we observe that $\int_G f_R \,\mathrm{d}\mu = (f)$. Moreover,

$$\|f_R\|_{\mathrm{L}^q(G)} = \frac{|(f)|}{R}\left(\int_G |\chi_{Q_R}|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}} = \frac{|(f)|}{R^{1-1/q}} \to 0, \quad \text{as } R \to \infty.$$

Note that for all $R > 0$ we have $f - f_R \in \mathrm{L}^q(G) \cap \widehat{W}^{-1,q}(G)$ by part (i). Since $f - f_R$ converges to f in $\mathrm{L}^q(G)$ as $R \to \infty$, the assertion is proven.

(iii) Concerning uniqueness, let $u \in \widehat{W}^{2,q}_\omega(G)$ be such that $-\Delta u = 0$. Then u and consequently also $\nabla^2 u \in \mathrm{L}^q_\omega(G)$ are harmonic. Lemma 4.5 shows $\nabla^2 u = 0$.
For existence, we note that we may assume $f \in C_0^\infty(G)$ by density. Moreover, by Corollary 4.10 and Remark 4.11 the assertion is true for all $f \in \mathrm{L}^q_\omega(G) \cap \widehat{W}^{-1,q}_\omega(G)$. Hence, an approximation procedure using part (ii) yields a solution $u \in \widehat{W}^{2,q}(G)$. Furthermore, for each $j \in \{1, \dots, n\}$ we have $\partial_j f \in \widehat{W}^{-1,q}(G) \cap \widehat{W}^{-1,q}_\omega(G)$. Since $\partial_j u \in \widehat{W}^{1,q}(G)$ is the unique weak solution to

$$(\nabla \partial_j u, \nabla \varphi) = [\partial_j f, \varphi], \quad \varphi \in C_0^\infty(G),$$

Proposition 4.7(ii) shows $\partial_j u \in \widehat{W}^{1,q}_\omega(G)$, whence we obtain $u \in \widehat{W}^{2,q}_\omega(G)$.

(iv) This is just another application of Proposition 4.7(ii).

□

4.2. Analysis of the Stokes Equations

Let $q \in (1, \infty)$ and $\omega \in A_q(G)$ and let us introduce the Banach spaces

$$X^q_\omega(G) := \widehat{W}^{1,q}_\omega(G)^n \times \mathrm{L}^q_\omega(G),$$
$$Y^q_\omega(G) := \widehat{W}^{-1,q}_\omega(G)^n \times \mathrm{L}^q_\omega(G),$$

equipped with the respective product space norms. Let $(f, g) \in Y^q_\omega(G)$. We are interested in weak solutions $(u, \mathfrak{p}) \in X^q_\omega(G)$ to the Stokes equations

$$\begin{cases} (\nabla u, \nabla \varphi) - (\mathfrak{p}, \operatorname{div} \varphi) = [f, \varphi], \quad \varphi \in C_0^\infty(G)^n, \\ \operatorname{div} u = g. \end{cases} \tag{4.5}$$

Note that the unique solvability of (4.5) is equivalent to saying that the linear and bounded operator $S_{q,\omega} : X^q_\omega(G) \to Y^q_\omega(G)$ defined via

$$S_{q,\omega}(u, \mathfrak{p}) := \begin{pmatrix} S'_{q,\omega}(u, \mathfrak{p}) \\ -\operatorname{div} u \end{pmatrix} \tag{4.6}$$

is an isomorphism of Banach spaces, where we have written

$$S'_{q,\omega}(u, \mathfrak{p}) := (\nabla u, \nabla \varphi) - (\mathfrak{p}, \operatorname{div} \varphi), \quad (u, \mathfrak{p}) \in X^q_\omega(G), \quad \varphi \in C_0^\infty(G)^n.$$

Lemma 4.13. *Let $q_i \in (1, \infty)$ and $\omega_i \in A_{q_i}(G)$, $i = 1, 2$, respectively. Assume that $(u, \mathfrak{p}) \in X^{q_1}_{\omega_1}(G)$ satisfies $\|S_{q_2,\omega_2}(u, \mathfrak{p})\|_{Y^{q_2}_{\omega_2}(G)} < \infty$. Then $(u, \mathfrak{p}) \in X^{q_2}_{\omega_2}(G)$ as well. Moreover, there is an $A_{q_2}(G)$-consistent constant $c = c(n, q_2, \omega_2) > 0$ such that*

$$\|(u, \mathfrak{p})\|_{X^{q_2}_{\omega_2}(G)} \le c \|S_{q_2,\omega_2}(u, \mathfrak{p})\|_{Y^{q_2}_{\omega_2}(G)}. \tag{4.7}$$

Proof. Choose the special test function $\varphi := \nabla w$, $w \in C_0^\infty(G)$ and compute with Lemma 4.5 (ii) and Corollary 4.4

$$\begin{aligned}\|\mathfrak{p}\|_{\mathrm{L}^{q_2}_{\omega_2}(G)} &= \sup_{0\neq w\in C_0^\infty(G)} \frac{|(\operatorname{div} u - \mathfrak{p}, \Delta w)|}{\|\Delta w\|_{\mathrm{L}^{q_2'}_{\omega_2'}(G)}} + \|\operatorname{div} u\|_{\mathrm{L}^{q_2}_{\omega_2}(G)} \\ &\leq c \sup_{0\neq w\in C_0^\infty(G)} \frac{|(\nabla u, \nabla^2 w) - (\mathfrak{p}, \Delta w)|}{\|\nabla^2 w\|_{\mathrm{L}^{q_2'}_{\omega_2'}(G)}} + \|\operatorname{div} u\|_{\mathrm{L}^{q_2}_{\omega_2}(G)} \leq c\|S_{q_2,\omega_2}(u,\mathfrak{p})\|_{Y^{q_2}_{\omega_2}(G)}.\end{aligned}$$

Furthermore, choosing another special test function $\varphi := \partial_j w$, where $w \in C_0^\infty(G)^n$ and $j \in \{1,\dots,n\}$, we obtain

$$\begin{aligned}\|\partial_j u\|_{\mathrm{L}^{q_2}_{\omega_2}(G)} &= \sup_{0\neq w\in C_0^\infty(G)^n} \frac{|(\partial_j u, \Delta w)|}{\|\Delta w\|_{\mathrm{L}^{q_2'}_{\omega_2'}(G)}} \\ &\leq c\|\mathfrak{p}\|_{\mathrm{L}^{q_2}_{\omega_2}(G)} + c \sup_{0\neq w\in C_0^\infty(G)^n} \frac{|(\nabla u, \nabla\partial_j w) - (\mathfrak{p}, \operatorname{div}\partial_j w)|}{\|\nabla^2 w\|_{\mathrm{L}^{q_2'}_{\omega_2'}(G)}} \\ &\leq c\|S_{q_2,\omega_2}(u,\mathfrak{p})\|_{Y^{q_2}_{\omega_2}(G)}.\end{aligned}$$

$A_{q_2'}(G)$-consistency (and hence $A_{q_2}(G)$-consistency) follows from Corollary 4.4. □

Theorem 4.14. *Let $q, q_i \in (1,\infty)$, $\omega \in A_q(G)$ and $\omega_i \in A_{q_i}(G)$, $i = 1, 2$, respectively.*

(i) *For all $(f, g) \in Y^q_\omega(G)$ there exists a unique solution $(u,\mathfrak{p}) \in X^q_\omega(G)$ to (4.5). Moreover, there is an A_q-consistent constant $c = c(n, q, \omega) > 0$ such that*

$$\|(u,\mathfrak{p})\|_{X^q_\omega(G)} \leq c\|(f,g)\|_{Y^q_\omega(G)}.$$

(ii) *If $(f,g) \in Y^{q_1}_{\omega_1}(G) \cap Y^{q_2}_{\omega_2}(G)$, then the unique solution $(u,\mathfrak{p}) \in X^{q_1}_{\omega_1}(G)$ to (4.5) satisfies also $(u,\mathfrak{p}) \in X^{q_2}_{\omega_2}(G)$.*

Proof. (i) Lemma 4.13 applied with exponents $q = q_1 = q_2$ and weights $\omega = \omega_1 = \omega_2$ shows that $S_{q,\omega}$ is injective and has closed range. Observe that $(X^q_\omega(G))' = Y^{q'}_{\omega'}(G)$ and that due to

$$\begin{aligned}[(u,\mathfrak{p}),(S_{q,\omega})'(v,\mathfrak{q})] &= [S_{q,\omega}(u,\mathfrak{p}),(v,\mathfrak{q})] \\ &= (\nabla u,\nabla v) - (\mathfrak{p},\operatorname{div} v) - (\operatorname{div} u,\mathfrak{q}) = [(u,\mathfrak{p}), S_{q',\omega'}(v,\mathfrak{q})],\end{aligned}$$

we have $(S_{q,\omega})' = S_{q',\omega'}$. Since $1 < q < \omega$ and $\omega \in A_q(G)$ were arbitrary, the closed range theorem yields that $S_{q,\omega}$ is an isomorphism.

(ii) By part (i), the unique solution $(u,\mathfrak{p}) \in X^{q_1}_{\omega_1}(G)$ fulfills

$$\|S_{q_2,\omega_2}(u,\mathfrak{p})\|_{Y^{q_2}_{\omega_2}(G)} = \|(f,g)\|_{Y^{q_2}_{\omega_2}(G)} < \infty.$$

Therefore, Lemma 4.13 shows that $(u,\mathfrak{p}) \in X^{q_2}_{\omega_2}(G)$.

□

Let us now consider strong solutions to the Stokes equations. To be more precise, we look at the problem

$$\begin{cases} -\Delta u + \nabla\mathfrak{p} = f, & \text{in } G, \\ \quad\nabla\operatorname{div} u = \nabla g, & \text{in } G. \end{cases} \tag{4.8}$$

Theorem 4.15. *Let $q, q_i \in (1,\infty)$, $\omega \in A_q(G)$ and $\omega_i \in A_{q_i}(G)$, $i = 1, 2$, respectively.*

(i) *For every $(f,g) \in \mathrm{L}^q_\omega(G)^n \times \widehat{W}^{1,q}_\omega(G)$ there is a unique solution $(u,\mathfrak{p}) \in \widehat{W}^{2,q}_\omega(G)^n \times \widehat{W}^{1,q}_\omega(G)$ to (4.8) satisfying*

$$\|\nabla^2 u, \nabla\mathfrak{p}\|_{\mathrm{L}^q_\omega(G)} \leq c\|f,\nabla g\|_{\mathrm{L}^q_\omega(G)},$$

where $c = c(n, q, \omega) > 0$ is an A_q-consistent constant.

(ii) *If* $f \in \mathrm{L}^{q_1}_{\omega_1}(G)^n \cap \mathrm{L}^{q_2}_{\omega_2}(G)^n$ *and* $g \in \widehat{W}^{1,q_1}_{\omega_1}(G) \cap \widehat{W}^{1,q_2}_{\omega_2}(G)$*, then the unique solution* $(u, \mathfrak{p}) \in \widehat{W}^{2,q_1}_{\omega_1}(G)^n \times \widehat{W}^{1,q_1}_{\omega_1}(G)$ *fulfills also the regularity* $(u, \mathfrak{p}) \in \widehat{W}^{2,q_2}_{\omega_2}(G)^n \times \widehat{W}^{1,q_2}_{\omega_2}(G)$.

Proof. (i) To prove uniqueness, let $(u, \mathfrak{p}) \in \widehat{W}^{2,q}_{\omega}(G)^n \times \widehat{W}^{1,q}_{\omega}(G)$ be a solution to (4.8) with data $(f, g) = (0, 0)$. Then $\nabla \operatorname{div} u = 0$, which shows that $\operatorname{div} u$ is constant. Therefore $\Delta \mathfrak{p} = 0$, and so $\mathfrak{p}$ and $\nabla \mathfrak{p} \in \mathrm{L}^q_\omega(G)$ are harmonic. In view of Lemma 4.5 we receive $\nabla \mathfrak{p} = 0$. It follows $\Delta u = 0$ and Corollary 4.12 gives $\nabla^2 u = 0$.

For existence, note that by Proposition 4.7 there is a unique pressure $\mathfrak{q} \in \widehat{W}^{1,q}_{\omega}(G)$ satisfying $\Delta_{q,\omega} \mathfrak{q} = \operatorname{div} f \in \widehat{W}^{-1,q}_{\omega}(G)$. Moreover, there is an A_q-consistent constant $c = c(n, q, \omega) > 0$ such that

$$\|\nabla \mathfrak{q}\|_{\mathrm{L}^q_\omega(G)} \leq c \|\operatorname{div} f\|_{\widehat{W}^{-1,q}_{\omega}(G)} \leq c\|f\|_{\mathrm{L}^q_\omega(G)}.$$

Define $\mathfrak{p} := \mathfrak{q} + g \in \widehat{W}^{1,q}_{\omega}(G)$. In view of Corollary 4.12 there is $u \in \widehat{W}^{2,q}_{\omega}(G)^n$ which is a solution to $-\Delta u = f - \nabla \mathfrak{p} \in \mathrm{L}^q_\omega(G)^n$ and satisfies

$$\|\nabla^2 u\|_{\mathrm{L}^q_\omega(G)} \leq c\|f - \nabla \mathfrak{p}\|_{\mathrm{L}^q_\omega(G)} \leq c\|f, \nabla g\|_{\mathrm{L}^q_\omega(G)},$$

where $c = c(n, q, \omega) > 0$ is an A_q-consistent. It remains to verify that $\nabla \operatorname{div} u = \nabla g$. Since $v := \nabla \operatorname{div} u - \nabla g \in \mathrm{L}^q_\omega(G)^n$ is harmonic, this is ensured by Lemma 4.5.

(ii) In the proof of part (i), the regularity of $\mathfrak{q}$ stems from Proposition 4.7 (i). Consequently, by Proposition 4.7 (ii) it follows $\mathfrak{q} \in \widehat{W}^{1,q_1}_{\omega_1}(G) \cap \widehat{W}^{1,q_2}_{\omega_2}(G)$ if $f \in \mathrm{L}^{q_1}_{\omega_1}(G)^n \cap \mathrm{L}^{q_2}_{\omega_2}(G)^n$. Similarly, Corollary 4.12 shows $u \in \widehat{W}^{2,q_1}_{\omega_1}(G)^n \cap \widehat{W}^{2,q_2}_{\omega_2}(G)^n$.

□

4.3. Proof of Theorem 1.4 in the Whole Space

In view of Proposition 4.7 we may define the pressure $\mathfrak{p} \in \widehat{W}^{1,q}_{\omega}(G)$ as the solution to the weak Laplace equation with right-hand side $\operatorname{div} f + (\lambda - \Delta) g \in \widehat{W}^{-1,q}_{\omega}(G)$. Moreover, let us define $v_g := \nabla W \in \mathrm{L}^q_\omega(G)^n$, where $W := \Delta^{-1}_{q,\omega} g$. Note that Corollary 4.10(i) implies $v_g \in W^{2,q}_\omega(G)^n$. Therefore, we can apply [22, Theorem 1] to solve

$$(\lambda - \Delta) v = f - (\lambda - \Delta) v_g - \nabla \mathfrak{p}.$$

Note that there is an A_q-consistent $c = c(n, q, \omega, \vartheta) > 0$ such that

$$\|\lambda v, \nabla^2 v\|_{\mathrm{L}^q_\omega(G)} \leq c\big(\|f\|_{\mathrm{L}^q_\omega(G)} + \|\nabla g\|_{\mathrm{L}^q_\omega(G)} + |\lambda| \|g\|_{\widehat{W}^{-1,q}_{\omega}(G)}\big).$$

Setting $u := v + v_g$, we obtain a solution $(u, \mathfrak{p}) \in W^{2,q}_\omega(G)^n \times \widehat{W}^{1,q}_{\omega}(G)$ to (1.2) with a corresponding A_q-consistent *a priori* estimate. This proves the existence part of the theorem.

For uniqueness and the additional regularity assertion, let $(u_1, \mathfrak{p}_1) \in W^{2,q_1}_{\omega_1}(G)^n \times \widehat{W}^{1,q_1}_{\omega_1}(G)$ and $(u_2, \mathfrak{p}_2) \in W^{2,q_2}_{\omega_2}(G)^n \times \widehat{W}^{1,q_2}_{\omega_2}(G)$ be the two corresponding solutions. Set $v := u_1 - u_2$ and $\mathfrak{q} := \mathfrak{p}_1 - \mathfrak{p}_2$. Then $\mathfrak{q}$ and hence also $\nabla \mathfrak{q} \in \mathrm{L}^{q_1}_{\omega_1}(G)^n + \mathrm{L}^{q_2}_{\omega_2}(G)^n$ is harmonic and by Lemma 4.5 we find $\nabla \mathfrak{q} = 0$. Consequently $(\lambda - \Delta) v = 0$ and thus $u = 0$.

5. The Half Space

5.1. Trace Spaces

It will be convenient to introduce the group $H := \mathbb{R}^{n_1 - 1} \times \mathbb{T}^{n_2}_L$, such that $G_+ = \mathbb{R}_+ \times H$. We usually use the symbol $'x$ to refer to an element in H. Note that doing so, we have several notations for a point $x \in G_+$ corresponding to the different splittings

$$G_+ = \mathbb{R}^{n_1}_+ \times \mathbb{T}^{n_2}_L = \mathbb{R}_+ \times H,$$

namely $x = (x', y) = (x_1, 'x)$. We introduce trace spaces in the weighted set-up as quotient spaces, identifying the boundary of G_+ with H. Note that we can introduce a differentiable structure on H similar to G, and consequently the spaces $C_0^\infty(H)$ and $\mathcal{S}(H)$ are well-defined.

Definition 5.1. Let $q \in (1,\infty)$, $\omega \in A_q(G)$ and $m \in \mathbb{N}$. Then we define the weighted trace spaces

$$T_\omega^{m,q}(H) := W_\omega^{m,q}(G_+)/\sim,$$
$$\widehat{T}_\omega^{m,q}(H) := \widehat{W}_\omega^{m,q}(G_+)/\sim,$$

where the equivalence relation identifies two functions whose difference has locally a vanishing trace. The topologies of $T_\omega^{m,q}(H)$ and $\widehat{T}_\omega^{m,q}(H)$ are given by the quotient topology. In particular, $T_\omega^{m,q}(H)$ and $\widehat{T}_\omega^{m,q}(H)$ are Banach spaces.

Remark 5.2. For $\phi \in T_\omega^{m,q}(H)$ we can choose $u \in W_\omega^{m,q}(G_+)$ with $[u] = \phi$. The norm of ϕ is given by

$$\|\phi\|_{T_\omega^{m,q}(H)} = \inf\{\|u - v\|_{W_\omega^{m,q}(G_+)} : \text{the trace of } v \in W_\omega^{m,q}(G_+) \text{ vanishes locally}\},$$

and this norm is independent of the choice of the respective representative $u \in W_\omega^{m,q}(G_+)$. We will write $\gamma(u) := [u]$ in the following. With this notation it is obvious that the trace operator $\gamma : W_\omega^{m,q}(G_+) \to T_\omega^{m,q}(H)$ is bounded, linear and surjective.

An analogous statement can be made in the case of homogeneous spaces, *i.e.*, about the trace operator $\gamma : \widehat{W}_\omega^{m,q}(G_+) \to \widehat{T}_\omega^{m,q}(H)$.

Remark 5.3. There are certain cases, in which the trace spaces can be identified with fractional order Sobolev spaces. For example, in the nonperiodic case $G_+ = \mathbb{R}_+^n$, it is well known that weights of the form $\omega_\alpha(x) := \operatorname{dist}(x, \partial\mathbb{R}_+^n)^\alpha$ are in the class $A_q(\mathbb{R}^n)$ for $\alpha \in (-1, q-1)$ and that $T_{\omega_\alpha}^{1,q}(\mathbb{R}^{n-1}) = W^{1-\frac{1+\alpha}{q},q}(\mathbb{R}^{n-1})$, see [18].

Lemma 5.4. *Let $q \in (1,\infty)$, $\omega \in A_q(G)$ and $m \in \mathbb{N}$. Then both $C_0^\infty(H)$ and $\mathcal{S}(H)$ can be viewed as subspaces of $T_\omega^{m,q}(H)$ and $\widehat{T}_\omega^{m,q}(H)$. More precisely, there is a bijection $\Gamma : C_0^\infty(H) \to \gamma(C_0^\infty(\overline{G_+}))$ such that the following diagram commutes*

$$\begin{tikzcd} C_0^\infty(\overline{G_+}) \arrow[r, "u\mapsto u(0,\cdot)"] \arrow[dr, "u\mapsto\gamma(u)"'] & C_0^\infty(H) \arrow[d, shift right, "\Gamma"'] \\ & \gamma(C_0^\infty(\overline{G_+})). \arrow[u, shift right, "\Gamma^{-1}"'] \end{tikzcd} \tag{5.1}$$

A similar statement holds true if we replace $C_0^\infty(H)$ and $\gamma(C_0^\infty(\overline{G_+}))$ by $\mathcal{S}(H)$ and $\gamma(\mathcal{S}(\overline{G_+}))$, respectively, where $\mathcal{S}(\overline{G_+}) := \{\phi|_{G_+} : \phi \in \mathcal{S}(G)\}$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(0) = 1$ be fixed. Note that for every $\phi \in C_0^\infty(H)$ it holds $E\phi \in C_0^\infty(\overline{G_+})$, where $E\phi(x_1, 'x) := \varphi(x_1)\phi('x)$. Hence $\gamma(E\phi) \in \gamma(C_0^\infty(\overline{G_+}))$, and we can define the bijection $\Gamma : C_0^\infty(H) \to \gamma(C_0^\infty(\overline{G_+}))$ by means of

$$\Gamma(\phi) := \gamma(E\phi) \quad \text{and} \quad \Gamma^{-1}(\gamma(u)) := u(0,\cdot),$$

where $\Gamma^{-1}(\gamma(u))$ is well-defined since for $u_1, u_2 \in C_0^\infty(\overline{G_+})$ with $\gamma(u_1) = \gamma(u_2)$ it holds by definition $u_1(0,\cdot) = u_2(0,\cdot)$. Then $\Gamma^{-1}\Gamma = \Gamma\Gamma^{-1} = \mathsf{id}$ and hence the diagram (5.1) commutes. Since $C_0^\infty(H)$ can be identified with $\gamma(C_0^\infty(\overline{G_+}))$, it follows

$$C_0^\infty(H) = \gamma(C_0^\infty(\overline{G_+})) \subset \gamma(W_\omega^{k,q}(G_+)) = T_\omega^{k,q}(G_+).$$

The assertion about the space $\mathcal{S}(H)$ and about homogeneous trace spaces follows analogously. $\square$

Proposition 5.5. *Let $k \in \mathbb{N}$, $q, q_i \in (1,\infty)$ and $\omega \in A_q(G)$, $\omega_i \in A_{q_i}(G)$, respectively, where $i = 1, 2$. The space $C_0^\infty(\overline{G_+})$ is dense in $\widehat{W}_{\omega_1}^{1,q_1}(G_+) \cap \widehat{W}_{\omega_2}^{1,q_2}(G_+)$, $\widehat{W}_{\omega_1}^{2,q_1}(G_+) \cap \widehat{W}_{\omega_2}^{2,q_2}(G_+)$ and $W_{\omega_1}^{k,q_1}(G_+) \cap W_{\omega_2}^{k,q_2}(G_+)$. Moreover, $W_\omega^{2,q}(G_+) = \mathrm{L}_\omega^q(G_+) \cap \widehat{W}_\omega^{2,q}(G_+)$.*

Proof. Follows from the corresponding results in G and the fact that for $N \in \mathbb{N}$, $i \in \{1, \dots, N\}$, $m_i \in \mathbb{N}_0$, $1 \le q_i < \infty$ and $\omega_i \in A_{q_i}(\mathbb{R}^n)$ there is an extension operator $\Lambda : \bigcap_{i=1}^N \widehat{W}^{m_i,q_i}_{\omega_i}(G_+) \to \bigcap_{i=1}^N \widehat{W}^{m_i,q_i}_{\omega_i}(G)$, see [6]. In fact, in [6] the assertion is proved only for $n_2 = 0$, but revising the proof, it is readily seen that also the general case is admissible. □

Lemma 5.6. *Let $q \in (1,\infty)$, $\omega \in A_q(G)$ and $k \in \mathbb{N}$. Then $C_0^\infty(H)$ is dense in $\widehat{T}^{1,q}_\omega(H)$, $\widehat{T}^{2,q}_\omega(H)$ and $T^{k,q}_\omega(H)$.*

Proof. By Lemma 5.4 it is justified to write $\gamma(C_0^\infty(\overline{G_+})) = C_0^\infty(H)$. Since we know by Proposition 5.5 that $C_0^\infty(\overline{G_+})$ is dense in $\widehat{W}^{1,q}_\omega(G_+)$, $\widehat{W}^{2,q}_\omega(G_+)$ and $W^{k,q}_\omega(G_+)$, respectively, the assertion follows since the trace operator γ is bounded in the respective spaces by Remark 5.2. □

Lemma 5.7. *Let $u \in W^{2,1}_{\mathrm{loc}}(\overline{G_+})$. Then for all $j \in \{2, \dots, n\}$ it holds $\gamma(\partial_j u) = \partial_j \gamma(u)$.*

Proof. Observe that $u \in W^{2,1}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+})$. By the proof of Lemma 3.4 in [10], we have $\partial_j \gamma_{\mathbb{R}^n_+}(u) = \gamma_{\mathbb{R}^n_+}(\partial_j u)$. This implies immediately $\gamma(\partial_j u) = \partial_j \gamma(u)$. □

Lemma 5.8. *Let $q \in (1,\infty)$, $\omega \in A_q(G)$, $\phi \in \widehat{T}^{2,q}_\omega(H)$ and $\psi \in T^{2,q}_\omega(H)$. Then it holds for all $j \in \{2, \dots, n\}$*

$$|\partial_j \phi|_{\widehat{T}^{1,q}_\omega(H)} \le |\phi|_{\widehat{T}^{2,q}_\omega(H)},$$
$$\|\partial_j \psi\|_{T^{1,q}_\omega(H)} \le \|\psi\|_{T^{2,q}_\omega(H)}.$$

Proof. Let $\varepsilon > 0$. By Remark 5.2, we can choose $u \in \widehat{W}^{2,q}_\omega(G_+)$ with $\gamma(u) = \phi$ and

$$\|\nabla^2 u\|_{\mathrm{L}^q_\omega(G_+)} \le (1+\varepsilon)|\phi|_{\widehat{T}^{2,q}_\omega(H)}.$$

Since $\gamma(\partial_j u) = \partial_j \gamma(u) = \partial_j \phi$ by Lemma 5.7, it follows

$$|\partial_j \phi|_{\widehat{T}^{1,q}_\omega(H)} \le \|\nabla \partial_j u\|_{\mathrm{L}^q_\omega(G_+)} \le \|\nabla^2 u\|_{\mathrm{L}^q_\omega(G_+)} \le (1+\varepsilon)|\phi|_{\widehat{T}^{2,q}_\omega(H)},$$

whence the result follows for $\phi \in \widehat{T}^{2,q}_\omega(H)$. The second assertion is proven similarly. □

Lemma 5.9. *Let $q \in (1,\infty)$, $\omega \in A_q(G)$, $u \in W^{1,q}_\omega(G_+)$ and let $v \in C_0^\infty(\overline{G_+})$. Then it holds for all $j \in \{1, \dots, n\}$*

$$\int_{G_+} u\, \partial_j v \, d\mu = -\int_{G_+} v\, \partial_j u \, d\mu - \delta_{1j} \int_H \gamma(u)\, \gamma(v)\, d\mu_H,$$

where δ_{1j} denotes the Kronecker delta.

Proof. By Lemma 5.6 and Proposition 5.5, we can assume $u \in C_0^\infty(\overline{G_+})$. Moreover $\partial_j(uv) = u\partial_j v + v\partial_j u$ and $\gamma(uv) = \gamma(u)\gamma(v)$. Hence it remains to show that

$$\int_{G_+} \partial_j w \, \mathrm{d}\mu = -\delta_{1j} \int_H \gamma(w)\, \mathrm{d}\mu_H$$

for $w \in C_0^\infty(\overline{G_+})$, which is standard. □

5.2. Weak Solutions to the Laplace Equation

Lemma 5.10. *Let $q \in (1,\infty)$, $\omega \in A_q(G)$ and $u \in \widehat{W}^{1,q}_{0,\omega}(G_+)$. Then for the zero extension*

$$Eu : G \to \mathbb{C}, \quad Eu(x) := \begin{cases} u(x), & \text{if } x \in G_+, \\ 0, & \text{else}, \end{cases}$$

it holds $Eu \in \widehat{W}^{1,q}_\omega(G)$.
Similarly, for $u \in W^{1,q}_{0,\omega}(G_+)$ it holds $Eu \in W^{1,q}_\omega(G)$.

Proof. It suffices to show the assertion for $u \in \widehat{W}^{1,q}_{0,\omega}(G_+)$, since trivially $Eu \in \mathrm{L}^q_\omega(G)$ for $u \in \mathrm{L}^q_\omega(G_+)$. Moreover, it suffices to prove that $\partial_i Eu$ coincides almost everywhere on G with the zero extension $E(\partial_i u)$ for all $i \in \{1, \dots, n\}$, since then

$$\|\partial_i Eu\|_{\mathrm{L}^q_\omega(G)} = \|E\partial_i u\|_{\mathrm{L}^q_\omega(G)} = \|\partial_i u\|_{\mathrm{L}^q_\omega(G_+)} < \infty.$$

So let $u \in \widehat{W}^{1,q}_{0,\omega}(G_+)$ and let B_ρ be a ball with radius $\rho \ll L$. Furthermore, let $\psi \in C_0^\infty(G)$ be such that $\psi = 1$ on $B_{\rho/2}$ and $\operatorname{supp}\psi \subset B_\rho$. Then $\psi u \in W_0^{1,r}(Q)$ for some $r > 1$, where $Q := B_\rho \cap G_+$. Take a sequence $\{u_m\}_{m\in\mathbb{N}} \subset C_0^\infty(Q)$ approximating ψu in the space $W_0^{1,r}(Q)$ and compute for every $\varphi \in C_0^\infty(B_{\rho/2})$

$$\begin{aligned}\int_G Eu\,\partial_i\varphi\,\mathrm{d}\mu &= \int_Q u\,\partial_i\varphi\,\mathrm{d}\mu = \lim_{m\to\infty}\int_Q u_m\,\partial_i\varphi\,\mathrm{d}\mu \\ &= -\lim_{m\to\infty}\int_Q \partial_i u_m\,\varphi\,\mathrm{d}\mu = -\int_Q \partial_i u\,\varphi\,\mathrm{d}\mu = -\int_G E(\partial_i u)\,\varphi\,\mathrm{d}\mu.\end{aligned}$$

Thus, $\partial_i Eu = E(\partial_i u)$ as an identity in $\mathcal{S}'(G)$ and the assertion is proven. □

Lemma 5.11. *Let $q_i \in (1,\infty)$ and $\omega_i \in A_q(G)$ for $i = 1, 2$.*

(i) *If $u \in \widehat{W}^{1,q_1}_{\omega_1}(G_+) + \widehat{W}^{1,q_2}_{\omega_2}(G_+)$ is harmonic on G_+ and $\gamma(u) = 0$, then $u = 0$.*

(ii) *If $u \in \widehat{W}^{1,q_1}_{\omega_1}(G_+) + \widehat{W}^{1,q_2}_{\omega_2}(G_+)$ satisfies $\gamma(u) = 0$ and $(\lambda - \Delta)u = 0$ in the sense of distributions for some $\lambda \in \mathbb{C} \setminus \overline{\mathbb{R}_-}$, then $u = 0$.*

Proof. By Lemma 3.2 we can assume $\omega_i = \omega_i^*$ for $i = 1, 2$. Let $\psi \in C_0^\infty(G)$ and set $\varphi := (\psi - \psi^*)|_{G_+} \in C_0^\infty(\overline{G_+})$. It follows $\gamma(\varphi) = 0$ and $\operatorname{supp}\varphi \subset Q$, where $Q = \overline{G_+ \cap U}$ for some smooth and compact $U \subset G$.

By Lemma 3.3, we know that there is $s > 1$ such that $u|_Q \in W^{1,s}(Q)$. Moreover, $\varphi \in W_0^{1,s'}(Q)$ and we thus find a sequence $\{\varphi_k\} \subset C_0^\infty(Q)$ converging to φ in $W_0^{1,s'}(Q)$. Let us denote by v the odd extension of u to the whole group. Then in the situation of part (i) it holds

$$\begin{aligned}\int_G v\,\Delta\psi\,\mathrm{d}\mu &= \int_{G_+} v\,\Delta\psi\,\mathrm{d}\mu + \int_{G_-} v\,\Delta\psi\,\mathrm{d}\mu = \int_{G_+} u\,\Delta(\psi - \psi^*)\,\mathrm{d}\mu \\ &= \int_Q u\,\Delta\varphi\,\mathrm{d}\mu \overset{\gamma(u)=0}{=} -\int_Q \nabla u\,\nabla\varphi\,\mathrm{d}\mu = -\lim_{k\to-\infty}\int_Q \nabla u\,\nabla\varphi_k\,\mathrm{d}\mu = 0,\end{aligned}$$

since u is harmonic on G_+. Therefore, $\int_G v\,\Delta\varphi\,\mathrm{d}\mu = 0$ for all $\varphi \in C_0^\infty(G)$ and Lemma 4.5 (ii) shows that v is harmonic. In particular, also ∇v is harmonic and so $\nabla v \in C^\infty(G)$. Moreover, we have

$$\|\nabla v\|_{\mathrm{L}^{q_1}_{\omega_1}(G)+\mathrm{L}^{q_2}_{\omega_2}(G)} = 2\|\nabla u\|_{\mathrm{L}^{q_1}_{\omega_1}(G_+)+\mathrm{L}^{q_2}_{\omega_2}(G_+)} < \infty,$$

Since v is smooth across the interface of G_+ and G_-, this implies the regularity $\nabla v \in \mathrm{L}^{q_1}_{\omega_1}(G) + \mathrm{L}^{q_2}_{\omega_2}(G)$. Lemma 4.5 (i) gives now $\nabla v = 0$, whence we conclude $u = 0$ by the boundary condition $\gamma(u) = 0$.

Part (ii) follows analogously. □

Lemma 5.12. *Let $q \in (1,\infty)$, $\omega \in A_q(G)$, $\vartheta \in (0,\pi)$ and $\lambda \in \Sigma_\vartheta$ with $|\lambda| = 1$.*

(i) *To all $\phi \in \widehat{T}^{1,q}_\omega(H)$ and all $F \in \widehat{W}^{-1,q}_\omega(G_+)$ there exists a unique solution $u \in \widehat{W}^{1,q}_\omega(G_+)$ to*

$$\begin{aligned}(\nabla u, \nabla\varphi) &= [F, \varphi], \quad \varphi \in \widehat{W}^{1,q'}_{0,\omega'}(G_+), \\ \gamma(u) &= \phi,\end{aligned} \tag{5.2}$$

and there is an A_q-consistent $c = c(n, q, \omega) > 0$ such that

$$\|\nabla u\|_{\mathrm{L}^q_\omega(G_+)} \le c\left(|\phi|_{\widehat{T}^{1,q}_\omega(H)} + \|F\|_{\widehat{W}^{-1,q}_\omega(G_+)}\right). \tag{5.3}$$

If $\phi \in \widehat{T}^{1;q_1}_{\omega_1}(H) \cap \widehat{T}^{1;q_2}_{\omega_2}(H)$ and $F \in \widehat{W}^{-1,q_1}_{\omega_1}(G_+) \cap \widehat{W}^{-1,q_2}_{\omega_2}(G_+)$, then the unique solution $u \in \widehat{W}^{1,q_1}_{\omega_1}(G_+)$ to (5.2) satisfies the regularity $u \in \widehat{W}^{1,q_1}_{\omega_1}(G_+) \cap \widehat{W}^{1,q_2}_{\omega_2}(G_+)$.

(ii) *To all $\phi \in T^{1,q}_{\omega}(H)$ and all $F \in W^{-1,q}_{\omega}(G_+)$ there exists $u \in W^{1,q}_{\omega}(G_+)$ such that*

$$\begin{aligned} \lambda(u,\varphi) + (\nabla u, \nabla\varphi) &= [F,\varphi], \quad \varphi \in W^{1,q'}_{0,\omega'}(G_+), \\ \gamma(u) &= \phi, \end{aligned} \tag{5.4}$$

and there is an A_q-consistent $c = c(n,q,\omega,\vartheta) > 0$ such that

$$\|u\|_{W^{1,q}_{\omega}(G_+)} \le c\big(\|\phi\|_{T^{1,q}_{\omega}(H)} + \|F\|_{W^{-1,q}_{\omega}(G_+)}\big).$$

If $\phi \in T^{1,q_1}_{\omega_1}(H) \cap T^{1,q_2}_{\omega_2}(H)$ and $F \in W^{-1,q_1}_{\omega_1}(G_+) \cap W^{-1,q_2}_{\omega_2}(G_+)$, then the unique solution $u \in W^{1,q_1}_{\omega_1}(G_+)$ to (5.4) satisfies the regularity $u \in W^{1,q_1}_{\omega_1}(G_+) \cap W^{1,q_2}_{\omega_2}(G_+)$.

Proof. (i) Assume for the moment $\phi = 0$. By Lemma 3.2 we can assume $\omega = \omega^*$. Thus, for every $\psi \in \widehat{W}^{1,q'}_{\omega'}(G)$ it holds $\varphi := (\psi - \psi^*)|_{G_+} \in \widehat{W}^{1,q'}_{0,\omega'}(G_+)$. Therefore, we can extend $F \in \widehat{W}^{-1,q}_{\omega}(G_+)$ to $f \in \widehat{W}^{-1,q}_{\omega}(G)$ by means of $[f,\psi] := [F,\varphi]$ for all $\psi \in \widehat{W}^{1,q'}_{\omega'}(G)$. Taking into consideration that $\omega = \omega^*$, we find

$$\begin{aligned} \|\nabla\psi\|_{\mathrm{L}^{q'}_{\omega'}(G)} &= \frac{1}{2}\left(\|\nabla\psi\|_{\mathrm{L}^{q'}_{\omega'}(G)} + \|\nabla\psi^*\|_{\mathrm{L}^{q'}_{\omega'}(G)}\right) \ge \frac{1}{2}\|\nabla(\psi-\psi^*)\|_{\mathrm{L}^{q'}_{\omega'}(G)} \\ &= \|\nabla(\psi-\psi^*)\|_{\mathrm{L}^{q'}_{\omega'}(G_+)} = \|\nabla\varphi\|_{\mathrm{L}^{q'}_{\omega'}(G_+)}, \end{aligned}$$

and consequently

$$\|f\|_{\widehat{W}^{-1,q}_{\omega}(G)} \le \|F\|_{\widehat{W}^{-1,q}_{\omega}(G_+)}.$$

Hence, we can employ Proposition 4.7 to find $v \in \widehat{W}^{1,q}_{\omega}(G)$ with $-\Delta_{q,\omega} v = f$ such that

$$\|\nabla v\|_{\mathrm{L}^q_{\omega}(G)} \le c\|f\|_{\widehat{W}^{-1,q}_{\omega}(G)},$$

where $c = c(n,q,\omega) > 0$ is A_q-consistent.
Note that $[f,\psi] = -[f,\psi^*]$, and so it holds for all $\psi \in \widehat{W}^{1,q'}_{\omega'}(G)$ the equality

$$-[\Delta_{q,\omega}(-v^*),\psi] = -(\nabla v^*, \nabla\psi) = -(\nabla v, \nabla\psi^*) = -[f,\psi^*] = [f,\psi].$$

This shows that $-v^* \in \widehat{W}^{1,q}_{\omega}(G)$ satisfies $-\Delta_{q,\omega}(-v^*) = f = -\Delta_{q,\omega} v$ as well. By uniqueness in $\widehat{W}^{1,q}_{\omega}(G)$, there is $K \in \mathbb{C}$ with $v = -v^* + K$, whence $\gamma(v) = K/2$. Defining $u := v|_{G_+} - K/2 \in \widehat{W}^{1,q}_{\omega}(G_+)$, we see that u satisfies (5.2) and (5.3) with $\phi = 0$.

Let now $\phi \in \widehat{T}^{1,q}_{\omega}(H)$ be arbitrary. By definition, we find a function $u_\phi \in \widehat{W}^{1,q}_{\omega}(G_+)$ with $\gamma(u_\phi) = \phi$ and $\|\nabla u_\phi\|_{\mathrm{L}^q_{\omega}(G_+)} \le 2|\phi|_{\widehat{T}^{1,q}_{\omega}(H)}$. Therefore, the problem is reduced to the situation with vanishing trace.

For uniqueness and the additional regularity assertion, let $u_i \in \widehat{W}^{1,q_i}_{\omega_i}(G_+)$, $i = 1,2$, denote the corresponding solutions to (5.2). Then $v := u_1 - u_2 \in \widehat{W}^{1,q_1}_{\omega_1}(G_+) + \widehat{W}^{1,q_2}_{\omega_2}(G_+)$ is harmonic with $\gamma(v) = 0$ and therefore $v = 0$ by Lemma 5.11.

(ii) Follows analogously.

□

Corollary 5.13. *Let $q \in (1,\infty)$ and $\omega \in A_q(G)$. Then $C_0^\infty(G_+)$ is dense in both $\widehat{W}^{1,q}_{0,\omega}(G_+)$ and $W^{1,q}_{0,\omega}(G_+)$.*

Proof. Let $F \in \widehat{W}^{-1,q'}_{\omega'}(G_+)$ satisfy $[F,\varphi] = 0$ for all $\varphi \in C_0^\infty(G_+)$. Then the weak solution $u \in \widehat{W}^{1,q'}_{0,\omega'}(G_+)$ to (5.2) with $\phi = 0$ is harmonic. Lemma 5.11 shows that $u = 0$. Therefore, $F = 0$ and the theorem of Hahn–Banach gives the assertion.

The second assertion follows analogously. □

Corollary 5.14. *Let $q, q_i \in (1,\infty)$ and $\omega \in A_q(G)$, $\omega_i \in A_q(G)$, $i = 1,2$.*

(i) *There is a linear, A_q-consistently bounded extension operator*

$$R : \widehat{T}^{1,q}_{\omega}(H) \to \widehat{W}^{1,q}_{\omega}(G_+),$$

with $\gamma R = \mathsf{id}_{\widehat{T}^{1,q}_{\omega}(H)}$ assigning to $\phi \in \widehat{T}^{1,q}_{\omega}(H)$ the unique solution to (5.2) with $F = 0$.

(ii) *It holds*

$$R : \widehat{T}^{1,q_1}_{\omega_1}(H) \cap \widehat{T}^{1,q_2}_{\omega_2}(H) \to \widehat{W}^{1,q_1}_{\omega_1}(G_+) \cap \widehat{W}^{1,q_2}_{\omega_2}(G_+).$$

Proof. Follows directly from Theorem 5.12. □

Corollary 5.15. *Let $q_i \in (1, \infty)$ and $\omega_i \in A_q(G)$, $i = 1, 2$. Then $C_0^\infty(H)$ is dense in $\widehat{T}^{1,q_1}_{\omega_1}(H) \cap \widehat{T}^{1,q_2}_{\omega_2}(H)$.*

Proof. Corollary 5.14 implies that

$$\gamma : \widehat{W}^{1,q_1}_{\omega_1}(G_+) \cap \widehat{W}^{1,q_2}_{\omega_2}(G_+) \to \widehat{T}^{1,q_1}_{\omega_1}(H) \cap \widehat{T}^{1,q_2}_{\omega_2}(H)$$

is surjective. Since it is also bounded, the assertion follows by Proposition 5.5. □

Corollary 5.16. *Let $q, q_i \in (1, \infty)$, $\omega \in A_q(G)$, $\omega_i \in A_q(G)$, $i = 1, 2$, $\vartheta \in (0, \pi)$ and $\lambda \in \Sigma_\vartheta$ with $|\lambda| = 1$.*

(i) *There is a linear, A_q-consistently bounded extension operator*

$$R_\lambda : T^{1,q}_{\omega}(H) \to W^{1,q}_{\omega}(G_+),$$

with $\gamma R_\lambda = \mathsf{id}_{T^{1,q}_{\omega}(H)}$, assigning to $\phi \in T^{1,q}_{\omega}(H)$ the unique solution to (5.4) with $F = 0$.

(ii) *It holds*

$$R_\lambda : T^{1,q_1}_{\omega_1}(H) \cap T^{1,q_2}_{\omega_2}(H) \to W^{1,q_1}_{\omega_1}(G_+) \cap W^{1,q_2}_{\omega_2}(G_+).$$

Proof. Follows directly from Theorem 5.12. □

We can even improve the regularity result about the extension operator R_λ. To do so, we need the following lemma.

Corollary 5.17. *If in the situation of Theorem 5.12(ii) we have additionally $F \in \mathrm{L}^q_\omega(G_+)$ and $\phi \in T^{2,q}_\omega(G_+)$, then $u \in W^{2,q}_\omega(G_+)$. Moreover, there is an A_q-consistent $c = c(n, q, \omega, \vartheta) > 0$ such that for all $j \in \{2, \dots, n\}$*

$$\|\partial_j u\|_{W^{1,q}_\omega(G_+)} \le c \left(\|\partial_j \phi\|_{T^{1,q}_\omega(H)} + \|\partial_j F\|_{W^{-1,q}_\omega(G_+)} \right).$$

Proof. It suffices to show $u \in W^{2,q}_\omega(G_+)$. If we can show this regularity result, Lemma 5.7 and Lemma 5.8 imply that $\gamma(\partial_j u) = \partial_j \phi \in T^{1,q}_\omega(H)$ for all $j \in \{2, \dots, n\}$ and the uniqueness assertion of Theorem 5.12 yields the claim.

Let $j \in \{2, \dots, n\}$ and let $U \in W^{2,q}_\omega(G_+)$ be such that $\gamma(U) = \phi$ with $\|U\|_{W^{2,q}_\omega(G_+)} \le 2\|\phi\|_{T^{2,q}_\omega(H)}$, which gives $\gamma(\partial_j U) = \partial_j \phi$ in view of Lemma 5.7. Note that due to Theorem 5.12, $u - U \in W^{1,q}_\omega(G_+)$ is the unique solution to (5.4) with boundary condition 0 and right-hand side

$$F - (\lambda - \Delta)U \in \mathrm{L}^q_\omega(G_+) \subset W^{-1,q}_\omega(G_+).$$

Due to Lemma 3.2 we can assume $\omega = \omega^*$. Hence, denote by $f, v \in \mathrm{L}^q_\omega(G)$ the odd extensions of $F \in \mathrm{L}^q_\omega(G_+)$ and $(\lambda - \Delta)U \in \mathrm{L}^q_\omega(G_+)$. Theorem 1 in [22] shows that there is a unique $w \in W^{2,q}_\omega(G)$ solving $(\lambda - \Delta)w = f - v$ on G and thus in particular on G_+. Since also $-w^* \in W^{2,q}_\omega(G)$ solves $(\lambda - \Delta)(-w^*) = f - v$, we obtain $\gamma(w) = 0$ and, by uniqueness, $w|_{G_+} = u - U$. Consequently, $u = w|_{G_+} + U \in W^{2,q}_\omega(G)$. □

Corollary 5.18. *Let $q_i \in (1, \infty)$ and $\omega_i \in A_{q_i}(G)$, $i = 1, 2$, $\vartheta \in (0, \pi)$ and $\lambda \in \Sigma_\vartheta$ with $|\lambda| = 1$. Then*

$$R_\lambda : T^{2,q_1}_{\omega_1}(H) \cap T^{2,q_2}_{\omega_2}(H) \to W^{2,q_1}_{\omega_1}(G_+) \cap W^{2,q_2}_{\omega_2}(G_+).$$

In particular, $C_0^\infty(H)$ is dense in $T^{2,q_1}_{\omega_1}(H) \cap T^{2,q_2}_{\omega_2}(H)$.

Proof. Let $\phi \in T^{2,q_1}_{\omega_1}(H) \cap T^{2,q_2}_{\omega_2}(H)$. For all $j \in \{2, \dots, n\}$, Corollary 5.17 applied to ϕ and $F := 0$ shows

$$R_\lambda : T^{2,q_1}_{\omega_1}(H) \cap T^{2,q_2}_{\omega_2}(H) \to W^{2,q_1}_{\omega_1}(G_+) \cap W^{2,q_2}_{\omega_2}(G_+).$$

It follows that

$$\gamma : W^{2,q_1}_{\omega_1}(G_+) \cap W^{2,q_2}_{\omega_2}(G_+) \to T^{2,q_1}_{\omega_1}(H) \cap T^{2,q_2}_{\omega_2}(H)$$

is surjective. Since γ is also bounded, the result follows from Proposition 5.5. □

In fact, the extension operators R and R_λ have a very familiar representation in terms of Poisson operators. To see this, we will first prove a preliminary lemma. In this lemma, we use Plancherel's theorem on the locally compact abelian group H. Note that the Fourier transform yields an isometry from $\mathrm{L}^2(H)$ to $\mathrm{L}^2(\hat{H})$ only if the corresponding Haar measures on H and $\hat{H}$ are normalized accordingly. In our case, this gives an additional $c_n > 0$ depending on the dimension n such that $\|f\|_{\mathrm{L}^2(H)} = c_n \|\mathcal{F}_H f\|_{\mathrm{L}^2(\hat{H})}$. However, as it turns out, we are only interested in finiteness of the L^2-norms, and we can therefore suppress the dimensional constant c_n in the following.

From now on we will use the abbreviation $s := |'\eta|$ for the Euclidean norm of $'\eta = (\eta_2, \dots, \eta_n) \in \hat{H}$.

Lemma 5.19. *Let $m, M : \hat{H} \to \mathbb{C}$ be measurable and of at most polynomial growth for $s \to \infty$. Assume that m and M/s are bounded for $s \to 0$. For $\psi \in \mathcal{S}(H)$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}_-$ with $|\lambda| = 1$, we write $\widehat{\psi} := \mathcal{F}_H \psi$ and define*

$$\begin{aligned} f_\lambda(x) &:= \mathcal{F}_H^{-1}[m e^{-x_1\sqrt{\lambda + s^2}} \widehat{\psi}]('x), \\ f(x) &:= \mathcal{F}_H^{-1}[m e^{-x_1 s} \widehat{\psi}]('x), \\ F(x) &:= \mathcal{F}_H^{-1}[M x_1 e^{-x_1 s} \widehat{\psi}]('x), \end{aligned}$$

where $x = (x_1, 'x) \in G_+$. Then $f_\lambda, f, F \in C^\infty(G_+)$ and for all $m \in \mathbb{N}_0$ it holds $f_\lambda, \nabla f, \nabla F \in W^{m,2}(G_+)$.

Proof. Once we have shown $\nabla F \in W^{m,2}(G_+)$ for all $m \in \mathbb{N}_0$, Sobolev's embedding theorem immediately gives $F \in C^\infty(G_+)$, and similarly for f and f_λ. Hence, we focus on the assertion about the $\mathrm{L}^2(G_+)$ regularity.

We want to employ Plancherel's theorem. Note that we are in the unweighted case, and hence we can write $\mathrm{L}^2(G_+) = \mathrm{L}^2(\mathbb{R}_+; \mathrm{L}^2(H))$. Observe $\widehat{\psi} := \mathcal{F}_H \psi \in \mathcal{S}(\hat{H})$. Thus, by elementary computation, we obtain

$$\begin{aligned} \|\partial_1 f\|^2_{\mathrm{L}^2(G_+)} &= \frac{1}{2} \int_{\hat{H}} s m^2 |\widehat{\psi}|^2 \,\mathrm{d}\mu_{\hat{H}} < \infty, \\ \|\partial_1 F\|^2_{\mathrm{L}^2(G_+)} &= \int_{\hat{H}} \frac{M^2}{4s} |\widehat{\psi}|^2 \,\mathrm{d}\mu_{\hat{H}} < \infty. \end{aligned}$$

Moreover, for $k \in \{2, \dots, n\}$, it holds

$$\begin{aligned} \|\partial_k f\|^2_{\mathrm{L}^2(G_+)} &= \frac{1}{2} \int_{\hat{H}} s m^2 \frac{\eta_k^2}{s^2} |\widehat{\psi}|^2 \,\mathrm{d}\mu_{\hat{H}} < \infty, \\ \|\partial_k F\|_{\mathrm{L}^2(G_+)} &= \int_{\hat{H}} \frac{M^2}{4s^2} \frac{\eta_k^2}{s} |\widehat{\psi}|^2 \,\mathrm{d}\mu_{\hat{H}} < \infty. \end{aligned}$$

Finally, since $\lambda \neq \mathbb{R}_-$, there is $\delta > 0$ such that $\mathrm{Re}(\sqrt{\lambda + s^2}) \geq \delta(1+s)$ for all $s > 0$, see for example the proof of [9, Lemma 2.5]. Therefore $|e^{-x_1\sqrt{\lambda+s^2}}| \leq e^{-x_1\delta(1+s)}$, and it follows

$$\|f_\lambda\|^2_{\mathrm{L}^2(G_+)} \leq \frac{1}{2\delta} \int_{\hat{H}} \frac{m^2}{1+s} |\widehat{\psi}|^2 \,\mathrm{d}\mu_{\hat{H}} < \infty.$$

Therefore $f_\lambda, \nabla f, \nabla F \in \mathrm{L}^2(G_+)$. To take care of the higher derivatives, note that for $k \in \{2, \dots, n\}$ it holds

$$\begin{aligned} \partial_k f_\lambda &:= \mathcal{F}_H^{-1} m e^{-x_1\sqrt{\lambda+s^2}} \mathcal{F}_H \widetilde{\psi}, & \partial_1 f_\lambda &:= \mathcal{F}_H^{-1} \widetilde{m}_\lambda e^{-x_1\sqrt{\lambda+s^2}} \mathcal{F}_H \psi, \\ \partial_k f &:= \mathcal{F}_H^{-1} m e^{-x_1 s} \mathcal{F}_H \widetilde{\psi}, & \partial_1 f &:= \mathcal{F}_H^{-1} \widetilde{m} e^{-x_1 s} \mathcal{F}_H \psi, \\ \partial_k F &:= \mathcal{F}_H^{-1} M x_1 e^{-x_1 s} \mathcal{F}_H \widetilde{\psi}, & \partial_1 F &:= \mathcal{F}_H^{-1} (\widetilde{m}_F + \widetilde{M} x_1) e^{-x_1 s} \mathcal{F}_H \psi, \end{aligned}$$

with $\widetilde{\psi} := \partial_k \psi \in \mathcal{S}(H)$, $\widetilde{m} := -sm$, $\widetilde{m}_\lambda := -m\sqrt{\lambda+s}$, $\widetilde{m}_F := M$ and $\widetilde{M} := -sM$. Observe that $\widetilde{m}$, $\widetilde{m}_\lambda$, $\widetilde{m}_F$ and $\widetilde{M}/s$ are bounded near the origin. Hence, in any case we are in one of the situations discussed above and it follows $f_\lambda, \nabla f, \nabla F \in W^{1,2}(G_+)$. Iterating this process, we can estimate every order of differentiability. $\square$

With these preparations in mind, we are ready to identify the Poisson operator with the extension operator R_λ. Let us denote by T_λ the linear operator defined on $\mathcal{S}(H)$ via

$$T_\lambda \phi(x) := \mathcal{F}_H^{-1} e^{-x_1\sqrt{\lambda+s^2}} \mathcal{F}_H \phi('x),$$

where $x = (x_1, 'x) \in G_+$. Note, that the case $\lambda = 0$ is included here. In accordance with our notation for R and R_λ, we will drop the index $\lambda = 0$ if no confusion can arise.

Theorem 5.20. *Let $q \in (1, \infty)$, $\omega \in A_q(G)$, $\vartheta \in (0, \pi)$, $\lambda \in \Sigma_\vartheta$ with $|\lambda| = 1$ and $\phi \in \mathcal{S}(H)$.*

(i) *There is an A_q-consistent constant $c = c(n, q, \omega) > 0$ such that*

$$\|\nabla T\phi\|_{\mathrm{L}^q_\omega(G_+)} \le c|\phi|_{\widehat{T}^{1,q}_\omega(H)},$$
$$\|\nabla^2 T\phi\|_{\mathrm{L}^q_\omega(G_+)} \le c|\phi|_{\widehat{T}^{2,q}_\omega(H)}.$$

In particular, $R : \widehat{T}^{1,q}_\omega(H) \to \widehat{W}^{1,q}_\omega(G_+)$ is the unique extension of the operator T to a bounded linear operator on $\widehat{T}^{1,q}_\omega(H)$ with the property $\gamma \circ R = \mathsf{id}_{\widehat{T}^{1,q}_\omega(H)}$. Moreover, it holds for almost all $x = (x_1, 'x) \in G_+$

$$R\phi(x) = \mathcal{F}_H^{-1} e^{-x_1 s} \mathcal{F}_H \phi('x), \quad \phi \in \widehat{T}^{1,q}_\omega(H).$$

(ii) *There is an A_q-consistent constant $c = c(n, q, \omega, \vartheta) > 0$ such that*

$$\|T_\lambda \phi\|_{W^{1,q}_\omega(G_+)} \le c\|\phi\|_{T^{1,q}_\omega(H)},$$
$$\|T_\lambda \phi\|_{W^{2,q}_\omega(G_+)} \le c\|\phi\|_{T^{2,q}_\omega(H)}.$$

In particular, $R_\lambda : T^{1,q}_\omega(H) \to W^{1,q}_\omega(G_+)$ is the unique extension of the operator T_λ to a bounded linear operator on $T^{1,q}_\omega(H)$ with the property $\gamma \circ R_\lambda = \mathsf{id}_{T^{1,q}_\omega(H)}$. Furthermore, it holds for almost all $x = (x_1, 'x) \in G_+$

$$R_\lambda \phi(x) = \mathcal{F}_H^{-1} e^{-x_1\sqrt{\lambda+s^2}} \mathcal{F}_H \phi('x), \quad \phi \in \mathrm{T}^{1,q}_\omega(H).$$

Proof. (i) Lemma 5.19 immediately yields $T\phi \in \widehat{W}^{1,2}(G_+) \cap C^\infty(G_+)$. The computation

$$\Delta T\phi = \sum_{i=1}^n \partial_i^2 \mathcal{F}_H^{-1} e^{-x_1 s} \hat{\phi} = \mathcal{F}_H^{-1} \sum_{j=2}^n (s^2 - \eta_j^2) e^{-x_1 s} \hat{\phi} = 0,$$

shows that $T\phi$ is harmonic.

Hence, for each $\phi \in \mathcal{S}(H)$, $T\phi \in \widehat{W}^{1,2}(G_+)$ is a solution to (5.2) with $F = 0$. Since $\mathcal{S}(H) \subset \widehat{T}^{1,2}(G) \cap \widehat{T}^{1,q}_\omega(G)$ for all $\omega \in A_q(G)$ by Corollary 5.6, the regularity assertion in Theorem 5.12 gives $T\phi = R\phi \in \widehat{W}^{1,q}_\omega(G)$ and an A_q-consistent constant $c = c(n, q, \omega)$ such that

$$\|\nabla T\phi\|_{\mathrm{L}^q_\omega(G)} = \|\nabla R\phi\|_{\mathrm{L}^q_\omega(G)} \le c|\phi|_{\widehat{T}^{1,q}_\omega(G)}.$$

Moreover, for $j \in \{2, \dots, n\}$ we have $\partial_j \phi \in \mathcal{S}(H)$, and hence by the same arguments it follows

$$\|\nabla \partial_j T\phi\|_{\mathrm{L}^q_\omega(G)} = \|\nabla T \partial_j \phi\|_{\mathrm{L}^q_\omega(G)} \le c|\partial_j \phi|_{\widehat{T}^{1,q}_\omega(G)} \le c|\phi|_{\widehat{T}^{2,q}_\omega(G)},$$

where we have used Lemma 5.8 in the last estimate. Since $T\phi$ is harmonic, we obtain $\partial_1^2 T\phi = -\sum_{j=2}^n \partial_j^2 T\phi$ and so

$$\|\partial_1^2 T\phi\|_{\mathrm{L}^q_\omega(G)} \leq (n-1)c|\phi|_{\widehat{T}^{2,q}_\omega(G)}.$$

Summarizing, we have proved the claimed *a priori* estimates.

Corollary 5.6 shows that $\mathcal{S}(H)$ is dense in $\widehat{T}^{1,q}_\omega(G)$ and therefore part (i) is proven.

(ii) Lemma 5.19 shows $T_\lambda\phi \in W^{1,2}(G_+) \cap C^\infty(G_+)$. Observe that $(\lambda - \Delta)T_\lambda\phi = 0$, since formally

$$\mathcal{F}_H(\lambda - \Delta) = (\lambda - \partial_1^2 + s^2)\mathcal{F}_H$$

and thus

$$\mathcal{F}_H(\lambda - \Delta)T_\lambda\phi = (\lambda - \partial_1^2 + s^2)e^{-x_1\sqrt{\lambda+s^2}}\hat{\phi} = 0.$$

Hence, $T_\lambda\phi \in W^{1,2}(G_+)$ is a solution to (5.4) with $F = 0$ and $\phi \in \mathcal{S}(H)$. Since $\mathcal{S}(H) \subset T^{1,2}(G) \cap T^{1,q}_\omega(G)$ for all $\omega \in A_q(G)$ by Corollary 5.6, the uniqueness assertion in Theorem 5.12 yields the assertion as in part (i).

□

5.3. Weak Solutions to the Stokes Equations

In this section, we investigate weak solutions to the Stokes equations in the periodic half space, *i.e.*, we consider the problem

$$\begin{cases} (\nabla u, \nabla\varphi) - (\mathfrak{p}, \operatorname{div}\varphi) = [f, \varphi], \quad \varphi \in C_0^\infty(G_+), \\ \operatorname{div} u = g, \\ \gamma(u) = \phi. \end{cases} \tag{5.5}$$

Lemma 5.21. *Let $q \in (1,\infty)$ and $\omega \in A_q(G)$. For $f = g = 0$ and $\phi \in \widehat{T}^{1,q}_\omega(H)^n$, there is a solution $(w, \mathfrak{q}) \in \widehat{W}^{1,q}_\omega(G_+)^n \times \mathrm{L}^q_\omega(G_+)$ to (5.5) satisfying*

$$\|\nabla w\|_{\mathrm{L}^q_\omega(G_+)} + \|\mathfrak{q}\|_{\mathrm{L}^q_\omega(G_+)} \leq c|\phi|_{\widehat{T}^{1,q}_\omega(H)}, \tag{5.6}$$

where $c = c(n,q,\omega) > 0$ is A_q-consistent.

Moreover, for $\phi \in \widehat{T}^{2,2}(H)^n$, this weak solution solves (5.5) even in a strong sense, i.e., *$(w, \mathfrak{q}) \in \widehat{W}^{2,2}(G_+)^n \times \widehat{W}^{1,2}(G_+)$ and there is a positive constant $c = c(n) > 0$ such that*

$$\|\nabla^2 w\|_{\mathrm{L}^2(G_+)} + \|\nabla\mathfrak{q}\|_{\mathrm{L}^2(G_+)} \leq c|\phi|_{\widehat{T}^{2,2}(H)}. \tag{5.7}$$

Proof. By Lemma 5.6, $\mathcal{S}(H)$ is dense in both $\widehat{T}^{1,q}_\omega(H)$ and $\widehat{T}^{2,2}(H)$, and so it suffices to construct a solution with the correct regularity and *a priori* estimate for $\phi \in \mathcal{S}(H)^n$. Hence, let $\phi \in \mathcal{S}(H)^n$ be fixed.

We define the pressure

$$\mathfrak{q} := -2\operatorname{div} R\phi = -2\sum_{i=1}^n \partial_i R\phi_i,$$

where R is the extension operator defined in Corollary 5.14. Then by Theorem 5.20 it follows that there is an A_q-consistent $c = c(n,q,\omega) > 0$ such that

$$\|\mathfrak{q}\|_{\mathrm{L}^q_\omega(G_+)} \leq \|\nabla R\phi\|_{\mathrm{L}^q_\omega(G_+)} \leq c|\phi|_{\widehat{T}^{1,q}_\omega(G_+)}. \tag{5.8}$$

Note that also $\|\mathfrak{q}\|_{\mathrm{L}^2(G_+)} \leq c'|\phi|_{\widehat{T}^{1,2}(G_+)}$ and hence $\nabla\mathfrak{q} \in \widehat{W}_0^{-1,2}(G_+)^n \cap \widehat{W}^{-1,q}_{0,\omega}(G_+)^n$. Moreover, we define for every $j \in \{1,\ldots,n\}$ the component w_j via

$$w_j(x) := \sum_{i=1}^n \mathcal{F}_H^{-1} m^{ij}_{x_1}\mathcal{F}_H\phi_i('x), \quad x = (x_1, 'x) \in G_+.$$

Here, the multipliers $m_{x_1}^{ij} \in \mathrm{L}^\infty(H)$, $i,j \in \{1,\dots,n\}$, are given by

$$\begin{aligned} m_{x_1}^{11}('\eta) &:= (1+x_1 s)e^{-x_1 s}, \\ m_{x_1}^{1j}('\eta) &:= m_{x_1}^{j1}('\eta) := -\mathrm{i}x_1\eta_j e^{-x_1 s}, \qquad && j \in \{2,\dots,n\}, \\ m_{x_1}^{ij}('\eta) &:= \left(\delta_{ij} - x_1 \tfrac{\eta_i\eta_j}{s}\right) e^{-x_1 s}, && i,j \in \{2,\dots,n\}, \end{aligned} \tag{5.9}$$

and $s := |'\eta|$. The definition of $m_{x_1}^{ij}('\eta)$ is only meaningful for $'\eta \neq 0$, but it should be understood that we define $m_{x_1}^{ij}(0) = \delta_{ij}$. Note that all coefficients in front of $e^{-x_1 s}$ in (5.9) are sums with summands of the form m or Mx_1, where $m, M : \hat{H} \to \mathbb{C}$ satisfy the conditions in Lemma 5.19. Thus, w is smooth and it holds $D^\alpha \nabla w \in \mathrm{L}^2(G_+)$ for all $\alpha \in \mathbb{N}_0^n$, in particular for $|\alpha| = 0$. Let us verify that $\gamma(w) = \phi$, $\operatorname{div} w = 0$ and $\Delta w = \nabla \mathfrak{q}$. Since ϕ and w are smooth, we can evaluate point-wise and obtain easily $\gamma(w) = \phi$ by considering $x_1 \searrow 0$. Concerning the divergence, we compute

$$\operatorname{div} w = \sum_{j=1}^n \sum_{i=1}^n \partial_j \mathcal{F}_H^{-1} m_{x_1}^{ij} \mathcal{F}_H \phi_i = \sum_{i=1}^n \sum_{j=1}^n \partial_j \mathcal{F}_H^{-1} m_{x_1}^{ij} \mathcal{F}_H \phi_i. \tag{5.10}$$

Thus, it suffices to show that the inner sum vanishes for each $i \in \{1,\dots,n\}$ separately. Let us apply $\mathcal{F}_H$ to (5.10) for notational convenience. Then, for $i = 1$ we obtain with $\hat{\phi}_1 := \mathcal{F}_H \phi_1$ and using (5.9)

$$\mathcal{F}_H \sum_{j=1}^n \partial_j \mathcal{F}_H^{-1} m_{x_1}^{1j} \hat{\phi}_1 = \partial_1 m_{x_1}^{11} \hat{\phi}_1 + \sum_{j=2}^n \mathrm{i}\eta_j m_{x_1}^{1j} \hat{\phi}_1 = x_1 e^{-x_1 s} \hat{\phi}_1 \left(\sum_{j=2}^n \eta_j^2 - s^2 \right) = 0.$$

Similarly, for $i \in \{2,\dots,n\}$,

$$\begin{aligned} \mathcal{F}_H \sum_{j=1}^n \partial_j \mathcal{F}_H^{-1} m_{x_1}^{ij} \hat{\phi}_i &= \partial_1 m_{x_1}^{i1} \hat{\phi}_i + \sum_{j=2}^n \mathrm{i}\eta_j m_{x_1}^{ij} \hat{\phi}_i \\ &= -\mathrm{i}\eta_i (1 - x_1 s) e^{-x_1 s} \hat{\phi}_i + \sum_{j=2}^n \mathrm{i}\eta_j \left(\delta_{ij} - x_1 \frac{\eta_i \eta_j}{s} \right) e^{-x_1 s} \hat{\phi}_i = 0. \end{aligned}$$

Thus, we have proven $\operatorname{div} w = 0$. It remains to show $\Delta w = \nabla \mathfrak{q}$, that is $\Delta w_j = \partial_j \mathfrak{q}$ for all $j \in \{1,\dots,n\}$. Again, we start with $j = 1$ and compute $\mathcal{F}_H \partial_1 \mathfrak{q}$, where we apply the Fourier transform merely for the sake of readability. By definition of $\mathfrak{q}$ and by the representation of R in Theorem 5.20,

$$\begin{aligned} \mathcal{F}_H \partial_1 \mathfrak{q} &= -2\mathcal{F}_H \partial_1 \sum_{i=1}^n \partial_i R \phi_i = -2\partial_1^2 \mathcal{F}_H R \phi_1 - 2 \sum_{i=2}^n i\eta_i \partial_1 \mathcal{F}_H R \phi_i \\ &= -2s^2 e^{-x_1 s} \hat{\phi}_1 + 2 \sum_{i=2}^n \mathrm{i}\eta_i s e^{-x_1 s} \hat{\phi}_i. \end{aligned} \tag{5.11}$$

On the other hand, $\mathcal{F}_H \Delta w_1 = \sum_{i=1}^n (\partial_1^2 - s^2) m_{x_1}^{i1} \hat{\phi}_i$. We have

$$\left(\partial_1^2 - s^2\right) m_{x_1}^{11} \hat{\phi}_1 = \left(\left(x_1 s^3 - s^2\right) - \left(s^2 + x_1 s^3\right) \right) e^{-x_1 s} \hat{\phi}_1 = -2s^2 e^{-x_1 s} \hat{\phi}_1,$$

and for $i \in \{2,\dots,n\}$

$$\left(\partial_1^2 - s^2\right) m_{x_1}^{i1} \hat{\phi}_i = \left(2\mathrm{i}\eta_i s - \mathrm{i}x_1 \eta_i s^2\right) e^{-x_1 s} \hat{\phi}_i + \mathrm{i}x_1 \eta_i s^2 e^{-x_1 s} \hat{\phi}_i = 2\mathrm{i}\eta_i s e^{-x_1 s} \hat{\phi}_i, \tag{5.12}$$

whence we see

$$\mathcal{F}_H \Delta w_1 = -2s^2 e^{-x_1 s} \hat{\phi}_1 + 2 \sum_{i=2}^n i\eta_i s e^{-x_1 s} \hat{\phi}_i. \tag{5.13}$$

Comparing (5.13) to (5.11), the relation $\Delta w_1 = \partial_1 \mathfrak{q}$ follows. To show $\Delta w_j = \partial_j \mathfrak{q}$ for $j \in \{2,\dots,n\}$, we proceed analogously. It holds

$$\mathcal{F}_H \partial_j \mathfrak{q} = 2i\eta_j s e^{-x_1 s} \hat{\phi}_1 + 2 \sum_{i=2}^n \eta_j \eta_i e^{-x_1 s} \hat{\phi}_i. \tag{5.14}$$

Since, as before, $\mathcal{F}_H \Delta w_j = \sum_{i=1}^n (\partial_1^2 - s^2) m_{x_1}^{ij} \hat{\phi}_i$, the calculation in (5.12) and

$$\begin{aligned}\left(\partial_1^2 - s^2\right) m_{x_1}^{ij} \hat{\phi}_i &= \left[\left(\delta_{ij} s^2 + 2\eta_i\eta_j - x_1\eta_i\eta_j s\right) - s^2 \left(\delta_{ij} - x_1 \frac{\eta_i\eta_j}{s}\right)\right] e^{-x_1 s}\hat{\phi}_i \\ &= 2\eta_i\eta_j e^{-x_1 s}\hat{\phi}_i.\end{aligned}$$

for $i \in \{2, \dots, n\}$ shows $\Delta w_j = \partial_j \mathfrak{q}$. In total, we have shown that indeed $\Delta w = \nabla \mathfrak{q}$ and $\operatorname{div} w = 0$. Since $\nabla \mathfrak{q} \in \widehat{W}_0^{-1,2}(G_+)^n \cap \widehat{W}_{0,\omega}^{-1,q}(G_+)^n$, the regularity assertion of Theorem 5.12 yields $w \in \widehat{W}_{\omega}^{1,q}(G_+)^n$ and an A_q-consistent constant $c = c(n, q, \omega)$ such that

$$\begin{aligned}\|\nabla w\|_{\mathrm{L}^q_\omega(G_+)} &\leq c \left(|\phi|_{\widehat{T}^{1,q}_\omega(H)} + \|\nabla \mathfrak{q}\|_{W^{-1,q}_{0,\omega}(G_+)}\right) \\ &\leq c \left(|\phi|_{\widehat{T}^{1,q}_\omega(H)} + \|\mathfrak{q}\|_{\mathrm{L}^q_\omega(G_+)}\right) \leq c|\phi|_{\widehat{T}^{1,q}_\omega(H)}.\end{aligned} \tag{5.15}$$

Together with (5.8), this yield the claimed *a priori* estimate (5.6).

It is left to show the *a priori* estimate (5.7). For $j \in \{2, \dots, n\}$ we know that $\partial_j w \in \widehat{W}^{1,2}(G_+)$ is a weak solution to the Laplace equation (5.2) with right hand side $\nabla \partial_j \mathfrak{q} \in \widehat{W}_0^{-1,2}(G_+)$ and with boundary data $\partial_j \phi \in \widehat{T}^{1,2}(H)$. Theorem 5.12 and Theorem 5.20 applied to $q = 2$ and $\omega = 1$ show

$$\begin{aligned}\|\nabla \partial_j w\|_{\mathrm{L}^2(G_+)} &\leq c \left(\|\nabla \partial_j \mathfrak{q}\|_{\widehat{W}_0^{-1,2}(G_+)} + |\partial_j \phi|_{\widehat{T}^{1,2}(H)}\right) \\ &\leq c \left(\|\nabla \mathfrak{q}\|_{\mathrm{L}^2(G_+)} + |\phi|_{\widehat{T}^{2,2}(H)}\right) \leq c|\phi|_{\widehat{T}^{2,2}(H)},\end{aligned}$$

with a constant $c > 0$. For the estimates of the derivatives with respect to the first variable we use the Stokes equations in order to obtain

$$\nabla \partial_1 w_1 = -\sum_{j=2}^n \nabla \partial_j w_j \quad \text{and} \quad \partial_1^2 w_j = \partial_j \mathfrak{q} - \sum_{i=2}^n \partial_i^2 w_j,$$

whence the estimate (5.7) follows. $\square$

Let $q \in (1, \infty)$ and $\omega \in A_q(G)$. Similarly as in the case of the periodic whole space G, we introduce the Banach spaces

$$\begin{aligned}X^q_\omega(G_+) &:= \widehat{W}^{1,q}_\omega(G_+)^n \times \mathrm{L}^q_\omega(G_+), \\ Y^q_\omega(G_+) &:= \widehat{W}^{-1,q}_\omega(G_+)^n \times \mathrm{L}^q_\omega(G_+) \times \widehat{T}^{1,q}_\omega(H)^n,\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}X^q_{0,\omega}(G_+) &:= \widehat{W}^{1,q}_{0,\omega}(G_+)^n \times \mathrm{L}^q_\omega(G_+), \\ Y^q_{0,\omega}(G_+) &:= \widehat{W}^{-1,q}_\omega(G_+)^n \times \mathrm{L}^q_\omega(G_+),\end{aligned}$$

furnished with the respective product space norms. With this notation, we have the following theorem.

Theorem 5.22. *Let $q, q_i \in (1, \infty)$, $\omega \in A_q(G)$ and $\omega_i \in A_{q_i}(G)$, $i = 1, 2$.*

(i) *For every $(f, g, \phi) \in Y^q_\omega(G_+)$, there is a unique $(u, \mathfrak{p}) \in X^q_\omega(G_+)$ solving the Stokes system (5.5) in a weak sense. Moreover, it holds*

$$\|(u, \mathfrak{p})\|_{X^q_\omega(G_+)} \leq c\|(f, g, \phi)\|_{Y^q_\omega(G_+)},$$

where $c = c(n, q, \omega) > 0$ is A_q-consistent.

(ii) *If $(f, g, \phi) \in Y^{q_1}_{\omega_1}(G_+) \cap Y^{q_2}_{\omega_2}(G_+)$, then the unique weak solution $(u, \mathfrak{p}) \in X^{q_1}_{\omega_1}(G_+)$ to (5.5) satisfies $(u, \mathfrak{p}) \in X^{q_2}_{\omega_2}(G_+)$.*

Proof. (i) By Hahn–Banach's theorem, $f \in \widehat{W}_{\omega}^{-1,q}(G_+)^n$ can be extended to a functional $f' \in \widehat{W}_{0,\omega}^{-1,q}(G_+)^n$ with same norm. Define $\bar{f} \in \widehat{W}_{\omega}^{-1,q}(G)^n$ by $[\bar{f}, \varphi] := [f', \varphi|_{G_+}]$ for all $\varphi \in \widehat{W}_{\omega'}^{1,q'}(G)^n$. Note that

$$\|\bar{f}\|_{\widehat{W}_{\omega}^{-1,q}(G)} \leq \|f'\|_{\widehat{W}_{0,\omega}^{-1,q}(G_+)} = \|f\|_{\widehat{W}_{\omega}^{-1,q}(G_+)}.$$

Moreover, denote by $\bar{g} \in \mathrm{L}_{\omega}^{q}(G)$ the zero extension of $g \in \mathrm{L}_{\omega}^{q}(G_+)$ to the whole group G. Then Theorem 4.14 gives a weak solution $(\bar{u}, \bar{\mathfrak{p}}) \in X_{\omega}^{q}(G)$ to (4.5) with data $(\bar{f}, \bar{g}) \in Y_{\omega}^{q}(G)$ satisfying the estimate

$$\|(\bar{u}, \bar{\mathfrak{p}})\|_{X_{\omega}^{q}(G)} \leq c\|(\bar{f}, \bar{g})\|_{Y_{\omega}^{q}(G)} \leq c\|(f, g)\|_{Y_{0,\omega}^{q}(G_+)}, \tag{5.17}$$

where $c = c(n, q, \omega) > 0$ is A_q-consistent. By Theorem 5.21, we find a solution $(w, \mathfrak{q}) \in X_{\omega}^{q}(G_+)$ to the Stokes equations on G_+ with data $(0, 0, \phi - \gamma(\bar{u}|_{G_+})) \in Y_{\omega}^{q}(G_+)$. Defining $u := \bar{u}|_{G_+} + w$ and $\mathfrak{p} := \bar{\mathfrak{p}}|_{G_+} + \mathfrak{q}$, we obtain a solution $(u, \mathfrak{p}) \in X_{\omega}^{q}(G_+)$ to (5.5) with a corresponding A_q-consistent estimate.

Concerning uniqueness, define the linear operator $S_{q,\omega} : X_{0,\omega}^{q}(G_+) \to Y_{0,\omega}^{q}(G_+)$ similar as in (4.6). It is immediate that $S_{q,\omega}$ is bounded and the considerations above with $\phi = 0$ show that it is surjective. Exactly as in the proof of Theorem 4.14, we see that $S_{q,\omega}$ is an isomorphism. In particular, let $(u, \mathfrak{p}) \in X_{\omega}^{q}(G_+)$ be a solution to (5.5) with data $(f, g, \phi) = (0, 0, 0)$. Then we have $(u, \mathfrak{p}) \in X_{0,\omega}^{q}(G_+)$ and thus it is justified to write $S_{q,\omega}(u, \mathfrak{p}) = (0, 0)$. Since $S_{q,\omega}$ is an isomorphism, it follows $(u, \mathfrak{p}) = (0, 0)$, which shows the uniqueness assertion.

(ii) The unique solution $(u, \mathfrak{p}) \in X_{\omega_1}^{q_1}(G_+)$ has by construction the form $u := \bar{u}|_{G_+} + w^1$ and $\mathfrak{p} := \bar{\mathfrak{p}}|_{G_+} + \mathfrak{q}^1$, where $(\bar{u}, \bar{\mathfrak{p}}) \in X_{\omega_1}^{q_1}(G)$ is the corresponding solution on the whole group G with respective data $(\bar{f}, \bar{g}) \in Y_{\omega_1}^{q_1}(G) \cap Y_{\omega_2}^{q_2}(G)$. By the regularity assertion in Theorem 4.14, we obtain $(\bar{u}, \bar{\mathfrak{p}}) \in X_{\omega_1}^{q_1}(G) \cap X_{\omega_2}^{q_2}(G)$. Moreover, the part $(w^1, \mathfrak{q}^1) \in X_{\omega_1}^{q_1}(G_+)$ is a solution to (5.5) with $f = g = 0$ and boundary data $\phi \in \widehat{T}_{\omega_1}^{1,q_1}(H) \cap \widehat{T}_{\omega_2}^{1,q_2}(H)$. Denote by $(w^2, \mathfrak{q}^2) \in X_{\omega_2}^{q_2}(G_+)$ the corresponding solution to (5.5) with the same boundary data. In virtue of Corollary 5.6 we find a sequence $\{\phi_k\}_{k\in\mathbb{N}} \subset \mathcal{S}(H)$ with $\phi_k \to \phi$ in $\widehat{T}_{\omega_1}^{1,q_1}(H) \cap \widehat{T}_{\omega_2}^{1,q_2}(H)$ as $k \to \infty$. Note that the corresponding solutions $(w_k, \mathfrak{q}_k)$ have been constructed explicitly in the proof of Lemma 5.21 and do not depend on $q \in (1, \infty)$ or $\omega \in A_q(G)$. Hence, for $i = 1, 2$ it holds

$$\begin{aligned} \|\nabla w_k - \nabla w^i\|_{\mathrm{L}_{\omega_i}^{q_i}(G_+)} &\to 0, \\ \|\mathfrak{q}_k - \mathfrak{q}^i\|_{\mathrm{L}_{\omega_i}^{q_i}(G_+)} &\to 0, \end{aligned}$$

as $k \to \infty$. By the uniqueness of the limit in the Hausdorff space $\mathrm{L}_{\omega_1}^{q_1}(G_+) + \mathrm{L}_{\omega_2}^{q_2}(G_+)$, it follows $(w^1, \mathfrak{q}^1) = (w^2, \mathfrak{q}^2) \in X_{\omega_2}^{q_2}(G_+)$.

□

5.4. Strong Solutions to the Stokes Equations

Let us consider strong solutions to the Stokes equations in the periodic half space. More precisely, we investigate the problem

$$\begin{cases} -\Delta u + \nabla \mathfrak{p} = f, & \text{in } G_+, \\ \nabla \operatorname{div} u = \nabla g, & \text{in } G_+, \\ \gamma(u) = \phi. \end{cases} \tag{5.18}$$

We have the following regularity result.

Lemma 5.23. *Let $q_i \in (1, \infty)$ and $\omega_i \in A_{q_i}(G)$, $i = 1, 2$. Assume furthermore $f \in \mathrm{L}_{\omega_1}^{q_1}(G_+) \cap \mathrm{L}_{\omega_2}^{q_2}(G_+)$, $g \in \widehat{W}_{\omega_1}^{1,q_1}(G_+) \cap \widehat{W}_{\omega_2}^{1,q_2}(G_+)$ and $\phi \in \widehat{T}_{\omega_1}^{2,q_1}(H) \cap \widehat{T}_{\omega_1}^{2,q_1}(H)$. If $(u, \mathfrak{p}) \in \widehat{W}_{\omega_1}^{2,q_1}(G_+) \times \widehat{W}_{\omega_1}^{1,q_1}(G_+)$ is a* *solution to* (5.18)*, then* $(u,\mathfrak{p}) \in \widehat{W}^{2,q_2}_{\omega_2}(G_+) \times \widehat{W}^{1,q_2}_{\omega_2}(G_+)$ *and there is an* $A_{q_2}(G)$*-consistent constant* $c = c(n, q_2, \omega_2) > 0$ *such that*

$$\|\nabla^2 u\|_{\mathrm{L}^{q_2}_{\omega_2}(G_+)} + \|\nabla \mathfrak{p}\|_{\mathrm{L}^{q_2}_{\omega_2}(G_+)} \leq c\left(\|f\|_{\mathrm{L}^{q_2}_{\omega_2}(G_+)} + \|\nabla g\|_{\mathrm{L}^{q_2}_{\omega_2}(G_+)} + |\phi|_{\widehat{T}^{2,q_2}_{\omega_2}(H)}\right).$$

Proof. Recall the definition of the spaces $X^q_\omega(G_+)$ and $Y^q_\omega(G_+)$ in (5.16). Let $j \in \{2,\dots,n\}$ and observe that $(\partial_j f, \partial_j g, \partial_j \phi) \in Y^{q_1}_{\omega_1}(G_+) \cap Y^{q_2}_{\omega_2}(G_+)$, where the regularity assertion for $\partial_j\phi$ stems from Lemma 5.8. Moreover, $(\partial_j u, \partial_j \mathfrak{p}) \in X^{q_1}_{\omega_1}(G_+)$ is a weak solution to the Stokes equations (5.5) with data $(\partial_j f, \partial_j g, \partial_j \phi)$. The regularity assertion in Theorem 5.22 gives $(\partial_j u, \partial_j \mathfrak{p}) \in X^{q_2}_{\omega_2}(G_+)$ and an $A_{q_2}(G)$-consistent $c = c(n, q_2, \omega_2) > 0$ such that

$$\begin{aligned}
&\|\nabla \partial_j u\|_{\mathrm{L}^{q_2}_{\omega_2}(G_+)} + \|\partial_j \mathfrak{p}\|_{\mathrm{L}^{q_2}_{\omega_2}(G_+)}\\
&\quad\leq c\left(\|\partial_j f\|_{\widehat{W}^{-1,q_2}_{\omega_2}(G_+)} + \|\partial_j g\|_{\mathrm{L}^{q_2}_{\omega_2}(G_+)} + |\partial_j \phi|_{\widehat{T}^{1,q_2}_{\omega_2}(H)}\right)\\
&\quad\leq c\left(\|f\|_{\mathrm{L}^{q_2}_{\omega_2}(G_+)} + \|\nabla g\|_{\mathrm{L}^{q_2}_{\omega_2}(G_+)} + |\phi|_{\widehat{T}^{2,q_2}_{\omega_2}(H)}\right).
\end{aligned}$$

For the derivatives with respect to the first variable, we use the Stokes equations (5.18) and observe for $k \in \{2,\dots,n\}$

$$\begin{aligned}
\nabla \partial_1 u_1 &= \nabla g - \textstyle\sum_{j=2}^n \nabla \partial_j u_j &&\in \mathrm{L}^{q_2}_{\omega_2}(G_+),\\
\partial_1^2 u_k &= f_k - \partial_k \mathfrak{p} - \textstyle\sum_{j=2}^n \partial_j^2 u_k &&\in \mathrm{L}^{q_2}_{\omega_2}(G_+),
\end{aligned} \tag{5.19}$$

which implies $u \in \widehat{W}^{2,q_2}_{\omega_2}(G_+)$, $\nabla \mathfrak{p} = f + \Delta u \in \mathrm{L}^{q_2}_{\omega_2}(G_+)$ and the full *a priori* estimate. □

Theorem 5.24. *Let* $q \in (1,\infty)$ *and assume* $\omega \in A_q(G)$*. For every* $f \in \mathrm{L}^q_\omega(G_+)$*,* $g \in \widehat{W}^{1,q}_\omega(G_+)$ *and* $\phi \in \widehat{T}^{2,q}_\omega(H)$ *there is a unique solution* $(u,\mathfrak{p}) \in \widehat{W}^{2,q}_\omega(G_+) \times \widehat{W}^{1,q}_\omega(G_+)$ *to* (5.18)*. Furthermore, there is an* A_q*-consistent constant* $c = c(n,q,\omega) > 0$ *such that*

$$\|\nabla^2 u\|_{\mathrm{L}^q_\omega(G_+)} + \|\nabla \mathfrak{p}\|_{\mathrm{L}^q_\omega(G_+)} \leq c\left(\|f\|_{\mathrm{L}^q_\omega(G_+)} + \|\nabla g\|_{\mathrm{L}^q_\omega(G_+)} + |\phi|_{\widehat{T}^{2,q}_\omega(H)}\right).$$

Proof. Concerning uniqueness, let $(u,\mathfrak{p}) \in \widehat{W}^{2,q}_\omega(G_+)^n \times \widehat{W}^{1,q}_\omega(G_+)$ be a solution to (5.18) with $(f,g,\phi) = (0,0,0)$. Then for $j \in \{2,\dots,n\}$, $(\partial_j u, \partial_j \mathfrak{p}) \in X^{q_1}_{\omega_1}(G_+)$ is a weak solution to the Stokes equations (5.5) with zero data. Hence $\nabla \partial_j u = 0$ and $\partial_j \mathfrak{p} = 0$. Plugging this into (5.19), we find also $\nabla \partial_1 u_1 = 0$ and $\partial_1^2 u_k = 0$ for all $k \in \{2,\dots,n\}$, and so $\nabla^2 u = 0$. Thus, also $\nabla \mathfrak{p} = \Delta u = 0$, and the uniqueness part is proven.

For existence, we may assume by density $f \in C_0^\infty(\overline{G_+})^n$, $g \in C_0^\infty(\overline{G_+})$ and $\phi \in C_0^\infty(H)^n$. Extend f by zero to $\bar{f} \in \mathrm{L}^2(G)^n$. Moreover, we use the extension operator of [6] to extend g to $\bar{g} \in \widehat{W}^{1,2}(G)$. Theorem 4.15 gives a corresponding solution $(\bar{u}, \bar{\mathfrak{p}}) \in \widehat{W}^{2,2}(G)^n \times \widehat{W}^{1,2}(G)$ to the Stokes equations (4.8) in the periodic whole space. Now Lemma 5.21 can be applied to find a solution $(w,\mathfrak{q}) \in \widehat{W}^{2,2}(G_+)^n \times \widehat{W}^{1,2}(G_+)$ to (5.18) with data $(0,0,\phi - \gamma(\bar{u}|_{G_+}))$. Defining $(u,\mathfrak{p}) = (\bar{u}|_{G_+} + w, \bar{\mathfrak{p}}|_{G_+} + \mathfrak{q})$, we have constructed a solution $(u,\mathfrak{p}) \in \widehat{W}^{2,2}(G_+)^n \times \widehat{W}^{1,2}(G_+)$ to (5.18). By Lemma 5.23 this solution is in $\widehat{W}^{2,q}_\omega(G_+)^n \times \widehat{W}^{1,q}_\omega(G_+)$ and there is an A_q-consistent $c = c(n,q,\omega) > 0$ such that

$$\|\nabla^2 u\|_{\mathrm{L}^q_\omega(G_+)} + \|\nabla \mathfrak{p}\|_{\mathrm{L}^q_\omega(G_+)} \leq c\left(\|f\|_{\mathrm{L}^q_\omega(G_+)} + \|\nabla g\|_{\mathrm{L}^q_\omega(G_+)} + |\phi|_{\widehat{T}^{2,q}_\omega(H)}\right).$$

□

5.5. Estimates on the Boundary

Define for $i \in \{2,\dots,n\}$ and $\phi \in \mathcal{S}(H)$ the Riesz transformation on H by

$$S_i\phi := \mathcal{F}_H^{-1} \frac{\mathrm{i}\eta_i}{s} \mathcal{F}_H \phi, \quad s = |'\eta|.$$

Moreover, define the operator

$$M_\lambda \phi := \mathcal{F}_H^{-1} \sqrt{\lambda + s^2} \mathcal{F}_H \phi$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}_-$. As usual, we will drop the index λ in the case $\lambda = 0$.

Lemma 5.25. *Let $q \in (1,\infty)$, $\omega \in A_q(G)$, $\phi \in \mathcal{S}(H)$ and $i,j \in \{2,\dots,n\}$. Then*

(i) $|S_i\phi|_{\widehat{T}^{1,q}_\omega(H)} \le c|\phi|_{\widehat{T}^{1,q}_\omega(H)}$ *and*
(ii) $\|\partial_j S_i \phi\|_{T^{1,q}_\omega(H)} \le c\|\phi\|_{T^{2,q}_\omega(H)}$.

In both cases, the constant $c = c(n,q,\omega) > 0$ is A_q-consistent.

Proof. (i) We define an extension of $S_i\phi$ from H to G_+ via

$$P_i\phi(x) := \mathcal{F}_H^{-1} \frac{\mathrm{i}\eta_i}{s} e^{-x_1 s} \mathcal{F}_H \phi('x), \quad x = (x_1, 'x) \in G_+.$$

It suffices to show that there is an A_q-consistent $c = c(n,q,\omega) > 0$ such that $\|\nabla P_i\phi\|_{\mathrm{L}^q_\omega(G_+)} \le c\|\nabla R\phi\|_{\mathrm{L}^q_\omega(G_+)}$, where R is the extension operator from Corollary 5.14. Indeed, since $P_i\phi$ is smooth by Lemma 5.19, we have $\gamma(P_i\phi) = S_i\phi$ and thus it follows from Theorem 5.12

$$|S_i\phi|_{\widehat{T}^{1,q}_\omega(H)} \le \|\nabla P_i\phi\|_{\mathrm{L}^q_\omega(G_+)} \le c\|\nabla R\phi\|_{\mathrm{L}^q_\omega(G_+)} \le c|\phi|_{\widehat{T}^{1,q}_\omega(H)}.$$

Observe that Theorem 5.20 shows $R\phi = \mathcal{F}_H^{-1} e^{-x_1 s} \mathcal{F}_H \phi$ whenever $\phi \in \mathcal{S}(H)$ and hence

$$\partial_1 P_i \phi = \mathcal{F}_H^{-1} \partial_1 \frac{\mathrm{i}\eta_i}{s} e^{-x_1 s} \mathcal{F}_H \phi = -\mathcal{F}_H^{-1} \mathrm{i}\eta_i e^{-x_1 s} \mathcal{F}_H \phi = -\partial_i R\phi.$$

Therefore we obtain $\|\partial_1 P_i\phi\|_{\mathrm{L}^q_\omega(G_+)} \le c\|\nabla R\phi\|_{\mathrm{L}^q_\omega(G_+)}$. For the derivatives $\partial_j P_i\phi$, we proceed as follows. First of all, we notice that in view of Lemma 3.2 we can assume $\omega = \omega^*$. Let us write $v := (E\partial_1 R\phi)^*$, where E denotes the extension of functions defined on G_+ by zero to the whole group G. Since $\omega = \omega^*$, we see

$$\|v\|_{\mathrm{L}^q_\omega(G)} = \|v^*\|_{\mathrm{L}^q_\omega(G)} = \|E\partial_1 R\phi\|_{\mathrm{L}^q_\omega(G)} \le \|\nabla R\phi\|_{\mathrm{L}^q_\omega(G_+)}. \tag{5.20}$$

Denote by $\mathcal{F}_\mathbb{R}$ the one-dimensional Fourier transform and observe $\mathcal{F}_\mathbb{R}\mathcal{F}_H = \mathcal{F}_G$. Assuming an appropriate normalization of the one-dimensional Lebesgue measure, for every fixed $r > 0$ it holds $\mathcal{F}_\mathbb{R} e^{-r|x_1|} = \frac{r}{\eta_1^2 + r^2}$. Since $2\int_0^\infty re^{-2tr}\,\mathrm{d}t = 1$ for all $r > 0$, we can make our key observation: For all $x_1 > 0$ and all $'\eta \in \hat{H}$ we can compute with $r = s$

$$\begin{aligned} e^{-x_1 s}\mathcal{F}_H\phi('\eta) &= 2e^{-x_1 s}\mathcal{F}_H\phi('\eta)\int_0^\infty se^{-2ts}\,\mathrm{d}t \\ &= -2\int_0^\infty e^{-(x_1+t)s}\mathcal{F}_H\partial_1 R\phi(t,'\eta)\,\mathrm{d}t \\ &= -2\int_\mathbb{R} e^{-|x_1+t|s}\mathcal{F}_H E\partial_1 R\phi(t,'\eta)\,\mathrm{d}t \\ &= -2\mathcal{F}_\mathbb{R}^{-1}\frac{s}{\eta_1^2+s^2}\mathcal{F}_\mathbb{R}\mathcal{F}_H v(x_1,'\eta). \end{aligned}$$

Note that $\eta_1^2 + s^2 = |\eta|^2$. Thus, for $x = (x_1, 'x) \in G_+$ we are led to

$$-\partial_j P_i\phi(x) = \mathcal{F}_H^{-1}\frac{\eta_j\eta_i}{s}e^{-x_1 s}\mathcal{F}_H\phi('x) = 2\mathcal{F}_G^{-1}\frac{\eta_j\eta_i}{|\eta|^2}\mathcal{F}_G v(x).$$

Hence, with Theorem 4.3 and estimate (5.20) we obtain an A_q-consistent constant $c = c(n,q,\omega) > 0$ such that

$$\|\partial_j P_i\phi\|_{\mathrm{L}^q_\omega(G_+)} \le c\|\nabla R\phi\|_{\mathrm{L}^q_\omega(G_+)}.$$

(ii) We have the trivial estimate $|\phi|_{\widehat{T}^{1,q}_\omega(H)} \leq \|\phi\|_{T^{2,q}_\omega(H)}$ and in view of Lemma 5.8 even $|\partial_j\phi|_{\widehat{T}^{1,q}_\omega(H)} \leq \|\partial_j\phi\|_{T^{2,q}_\omega(H)} \leq \|\phi\|_{T^{2,q}_\omega(H)}$. Moreover, it holds

$$\gamma(P_i\partial_j\phi) = S_i(\partial_j\phi) = -\mathcal{F}_H^{-1}\frac{\eta_j\eta_i}{s}\mathcal{F}_H\phi = \partial_j S_i\phi.$$

In view of part (i), we can compute for $k \in \{2,\dots,n\}$

$$\begin{aligned}
\|P_i\partial_j\phi\|_{\mathrm{L}^q_\omega(G_+)} &= \|\partial_j P_i\phi\|_{\mathrm{L}^q_\omega(G_+)} \leq c|\phi|_{\widehat{T}^{1,q}_\omega(H)} \leq c\|\phi\|_{T^{2,q}_\omega(H)},\\
\|\partial_1 P_i\partial_j\phi\|_{\mathrm{L}^q_\omega(G_+)} &= |\partial_i R\partial_j\phi|_{T^{1,q}_\omega(H)} \leq c|\partial_j\phi|_{\widehat{T}^{1,q}_\omega(H)} \leq c\|\phi\|_{T^{2,q}_\omega(H)},\\
\|\partial_k P_i\partial_j\phi\|_{\mathrm{L}^q_\omega(G_+)} &\leq c|\partial_j\phi|_{T^{1,q}_\omega(H)} \leq c\|\phi\|_{T^{2,q}_\omega(H)},
\end{aligned}$$

where $c = c(n,q,\omega) > 0$ is an A_q-consistent constant. Thus, we have proven $\|\partial_j S_i\phi\|_{T^{1,q}_\omega(H)} \leq \|P_i\partial_j\phi\|_{W^{1,q}_\omega(G_+)} \leq c\|\phi\|_{T^{2,q}_\omega(H)}$.

□

Lemma 5.26. *Let $q \in (1,\infty)$, $\omega \in A_q(G)$, $\vartheta \in (0,\pi)$, $\lambda \in \Sigma_\vartheta$ with $|\lambda| = 1$ and $\phi \in \mathcal{S}(H)$. Then*

(i) $\|M\phi\|_{T^{1,q}_\omega(H)} \leq c\|\phi\|_{T^{2,q}_\omega(H)}$, *where $c = c(n,q,\omega) > 0$ is an A_q-consistent constant, and*
(ii) $\|M_\lambda\phi\|_{T^{1,q}_\omega(H)} \leq c\|\phi\|_{T^{2,q}_\omega(H)}$, *where $c = c(n,q,\omega,\vartheta) > 0$ is an A_q-consistent constant.*

Proof. (i) We extend $M\phi$ from H to G_+ via

$$v(x) := \mathcal{F}_H^{-1} s e^{-x_1 s}\mathcal{F}_H\phi('x), \quad x = (x_1, 'x) \in G_+.$$

Since $\gamma(v) = M\phi$ and $v = -\partial_1 R\phi$, Theorem 5.20 shows that there is an A_q-consistent $c = c(n,q,\omega) > 0$ such that

$$\|M\phi\|_{T^{1,q}_\omega(H)} \leq \|v\|_{W^{1,q}_\omega(G_+)} \leq c\|\phi\|_{T^{2,q}_\omega(H)}.$$

(ii) Analogous to part (i).

□

5.6. Proof of Theorem 1.4 in the Half Space

We divide the proof into five steps.

Step 1: Scaling Argument

We claim that it is sufficient to prove the theorem for $\lambda \in \Sigma_\vartheta$ with $|\lambda| = 1$. For $\varepsilon > 0$, we write $\psi_\varepsilon(x) := \psi(x/\varepsilon)$ for a generic function ψ. Observe that $\omega \in A_q(G)$ if and only if $\omega_\varepsilon \in A_q(G_\varepsilon)$ with $\mathcal{A}_q(\omega_\varepsilon) = \mathcal{A}_q(\omega)$, where the locally compact abelian group $G_\varepsilon := \mathbb{R}^{n_1} \times \mathbb{T})\varepsilon L^{n_2}$ is equipped with the Haar measure μ_ε defined via

$$\int_{G_\varepsilon} f \,\mathrm{d}\mu_\varepsilon := \frac{1}{(\varepsilon L)^{n_2}}\int_{[0,\varepsilon L)}\int_{\mathbb{R}^{n-1}} f(x', x_n)\,\mathrm{d}x'\,\mathrm{d}y.$$

Note that the length of periodicity $L > 0$ does not enter in the constant of the transference principle of Theorem 4.1, and hence all results obtained so far hold true also in G_ε with estimates independent of $\varepsilon > 0$.

Define $G_{\varepsilon,+} := \mathbb{R}^{n_1}_+ \times \mathbb{T}^{n_2}_{\varepsilon L}$ and write $\lambda \in \Sigma_\vartheta$ in the form $\lambda = re^{i\varphi}$, where $r > 0$ and $\varphi \in (0, \vartheta + \frac{\pi}{2})$. For notational convenience, we set $\varepsilon := \sqrt{r}$. Assume that $(u,\mathfrak{p}) \in W^{2,q}_\omega(G_+)^n \times \widehat{W}^{1,q}_\omega(G_+)$ is a solution to the Stokes resolvent problem (1.2). We consider

$$(\tilde{u},\tilde{\mathfrak{p}}) \in W^{2,q}_{\omega_\varepsilon}(G_{\varepsilon,+})^n \times \widehat{W}^{1,q}_{\omega_\varepsilon}(G_{\varepsilon,+})$$

defined via $\tilde{u} := \varepsilon^2 u_\varepsilon$ and $\tilde{\mathfrak{p}} := \varepsilon \mathfrak{p}_\varepsilon$. Note that $(\tilde{u}, \tilde{\mathfrak{p}})$ solve

$$\begin{cases} e^{i\varphi}\tilde{u}(x) - \Delta\tilde{u}(x) + \nabla\tilde{\mathfrak{p}}(x) = f_\varepsilon(x), & \text{in } G_+, \\ \operatorname{div}\tilde{u}(x) = \varepsilon g_\varepsilon(x), & \text{in } G_+, \\ \gamma(\tilde{u}) = 0. \end{cases}$$

Also note that $\|f_\varepsilon\|_{L^q_{\omega_\varepsilon}(G_{\varepsilon,+})} = \varepsilon^{\frac{n-1}{q}}\|f\|_{\mathrm{L}^q_\omega(G_+)}$, $\|\nabla\varepsilon g_\varepsilon\|_{L^q_{\omega_\varepsilon}(G_{\varepsilon,+})} = \varepsilon^{\frac{n-1}{q}}\|\nabla g\|_{\mathrm{L}^q_\omega(G_+)}$, and $\|\varepsilon g_\varepsilon\|_{\widehat{W}^{-1,q}_{0,\omega_\varepsilon}(G_{\varepsilon,+})}$ $= \varepsilon^{\frac{n-1}{q}+2}\|g\|_{\widehat{W}^{-1,q}_{0,\omega}(G_+)}$. If we can show the resolvent estimate (1.4) for all $\tilde{\lambda} \in \Sigma_\vartheta$, $|\tilde{\lambda}| = 1$, with an A_q-consistent $c = c(n, q, \omega_\varepsilon, \vartheta)$ (and in particular independent of the length of periodicity $\varepsilon L > 0$), we obtain

$$\begin{aligned} \|\lambda u, \nabla^2 u, \nabla\mathfrak{p}\|_{\mathrm{L}^q_\omega(G_+)} &= \varepsilon^{-\frac{n-1}{q}}\|\tilde{u}, \nabla^2\tilde{u}, \nabla\tilde{\mathfrak{p}}\|_{L^q_{\omega_\varepsilon}(G_{\varepsilon,+})} \\ &\le c\varepsilon^{-\frac{n-1}{q}}\left(\|f_\varepsilon\|_{L^q_{\omega_\varepsilon}(G_{\varepsilon,+})} + \|\nabla\varepsilon g_\varepsilon\|_{L^q_{\omega_\varepsilon}(G_{\varepsilon,+})} + \|\varepsilon g_\varepsilon\|_{\widehat{W}^{-1,q}_{0,\omega_\varepsilon}(G_{\varepsilon,+})}\right) \\ &= c\left(\|f\|_{\mathrm{L}^q_\omega(G_+)} + \|\nabla g\|_{\mathrm{L}^q_\omega(G_+)} + |\lambda|\|g\|_{\widehat{W}^{-1,q}_{0,\omega}(G_+)}\right), \end{aligned}$$

where $c = c(n, q, \omega_\varepsilon, \vartheta) = c(n, q, \omega, \vartheta)$ is A_q-consistent, since $\mathcal{A}_q(\omega) = \mathcal{A}_q(\omega_\varepsilon)$. Hence, we can focus on the case $|\lambda| = 1$ and $\varepsilon L > 0$ in the following. Obviously, since $L > 0$ has been arbitrary all along, we can continue assuming $\varepsilon = 1$.

Step 2: Solution Formula

Lemma 3.2 shows that we can assume $\omega = \omega^*$. We split $f \in \mathrm{L}^q_\omega(G_+)^n$ by means of $f = (f_1, {}'f)$. Denote by $\bar{f}_1 \in \mathrm{L}^q_\omega(G)$ and ${}'\bar{f} \in \mathrm{L}^q_\omega(G)^{n-1}$ the odd extension of f_1 and the even extension of ${}'f$ to G, respectively, such that $\bar{f} := (\bar{f}_1, {}'\bar{f}) \in \mathrm{L}^q_\omega(G)^n$. Similarly, $\bar{g} \in W^{1,q}_\omega(G) \cap \widehat{W}^{-1,q}_\omega(G)$ denotes the even extension of $g \in W^{1,q}_\omega(G_+) \cap \widehat{W}^{-1,q}_{0,\omega}(G_+)$ to G. The whole space result of Theorem 1.4 yields a solution $(\bar{u}, \bar{\mathfrak{p}}) \in W^{2,q}_\omega(G)^n \times \widehat{W}^{1,q}_\omega(G)$ with data $\bar{f}$ and $\bar{g}$ and an A_q-consistent constant $c = c(n, q, \omega, \vartheta) > 0$ such that

$$\begin{aligned} \|\bar{u}, \nabla^2\bar{u}, \nabla\bar{\mathfrak{p}}\|_{\mathrm{L}^q_\omega(G_+)} &\le c\left(\|\bar{f}, \nabla\bar{g}\|_{\mathrm{L}^q_\omega(G)} + \|\bar{g}\|_{\widehat{W}^{-1,q}_\omega(G)}\right) \\ &\le 2c\left(\|f, \nabla g\|_{\mathrm{L}^q_\omega(G_+)} + \|g\|_{\widehat{W}^{-1,q}_{0,\omega}(G_+)}\right), \end{aligned} \tag{5.21}$$

where the last estimate is justified by the assumption $\omega = \omega^*$. Observe that also $(\bar{w}, \bar{\mathfrak{p}}^*) \in W^{2,q}_\omega(G)^n \times \widehat{W}^{1,q}_\omega(G)$ is a solution, where $\bar{w} := (-\bar{u}_1^*, \bar{u}')$. By uniqueness, we conclude $\bar{u}_1 = -\bar{u}_1^*$ and hence $\gamma(\bar{u}_1|_{G_+}) = 0$.

Set $\phi' := \gamma(\bar{u}'|_{G_+}) \in T^{2,q}_\omega(H)^{n-1}$. Then estimate (5.21) yields an A_q-consistent $c = c(n, q, \omega, \vartheta) > 0$ such that

$$\|\phi'\|_{T^{2,q}_\omega(H)} \le 2c\left(\|f, \nabla g\|_{\mathrm{L}^q_\omega(G_+)} + \|g\|_{\widehat{W}^{-1,q}_{0,\omega}(G_+)}\right). \tag{5.22}$$

If we subtract $(\bar{u}|_{G_+}, \bar{\mathfrak{p}}|_{G_+})$ from the resolvent problem (1.2), we obtain the reduced resolvent problem

$$\begin{cases} \lambda v - \Delta v + \nabla\mathfrak{q} = 0, & \text{in } G_+, \\ \operatorname{div} v = 0, & \text{in } G_+, \\ \gamma(v) = \phi := (0, \phi'), & \text{on } H, \end{cases}$$

which, after applying the Fourier transform $\mathcal{F}_H$, turns into an ordinary differential equation in x_1 of the form

$$\begin{cases} (\lambda - \partial_1^2 + s^2)\mathcal{F}_H v(x_1, {}'\eta) + \mathrm{i}\eta\mathcal{F}_H\mathfrak{q}(x_1, {}'\eta) = 0, \\ \partial_1 v_1(x_1, {}'\eta) + \sum_{j=2}^n \mathrm{i}\eta_j\mathcal{F}_H v_j(x_1, {}'\eta) = 0, \\ \mathcal{F}_H v(0, {}'\eta) = \mathcal{F}_H\phi({}'\eta), \end{cases}$$

for all $'\eta \in \hat{H}$. A solution to this ordinary differential equation can be found in [9] and [10] and is given by

$$(\mathcal{F}_H v(x_1, '\eta), \mathcal{F}_H \mathfrak{q}(x_1, '\eta)), \tag{5.23}$$

with

$$\begin{aligned} v_1(x) &:= \frac{1}{\lambda} \sum_{j=2}^{n} \left[\partial_j RM\phi_j - \partial_j R_\lambda M_\lambda \phi_j + \partial_j R_\lambda M\phi_j - \partial_j RM_\lambda \phi_j\right], \\ v_j(x) &:= \frac{1}{\lambda} \sum_{k=2}^{n} \left[\partial_k \partial_j R\phi_j - \partial_k \partial_j R_\lambda \phi_j + \lambda R_\lambda \phi_j \right. \\ &\qquad \left. + \partial_1 R_\lambda(\partial_k S_j \phi_j) + \partial_k R(S_j M_\lambda \phi_j)\right], \\ \mathfrak{q}(x) &:= -\mathcal{F}_H^{-1} \frac{1}{s^2}(\lambda + s^2 - \partial_1^2)\mathcal{F}_H \partial_1 v_1(x), \end{aligned}$$

where $j \in \{2, \dots, n\}$. The expressions for $\mathfrak{q}$, v_1 and v_j in this form can be found in [10, Formulae (29), (31) and (38)], respectively.

Step 3: Weighted Estimates

In order to prove estimates of the solution given in Step 2, we first assume $\phi' \in \mathcal{S}(H)^{n-1}$. Then it is an immediate consequence from Lemma 5.19 that

$$v \in W^{2,2}(G_+)^n, \text{ and hence also } \nabla\mathfrak{q} = -(\lambda - \Delta)v \in \mathrm{L}^2(G_+)^n.$$

Moreover, Theorem 5.20 on the Poisson operators R and R_λ, Lemma 5.25 on the trace Riesz operator S_j and Lemma 5.26 on the trace multiplication operators M and M_λ show, that there is an A_q-consistent constant $c = c(n, q, \omega, \vartheta)$ such that

$$\|v\|_{\mathrm{L}^q_\omega(G_+)} \le c\|\phi'\|_{T^{2,q}_\omega(H)}. \tag{5.24}$$

Since $(v, \mathfrak{q}) \in W^{2,2}(G_+)^n \times \widehat{W}^{1,2}(G_+)$ solves the Stokes problem

$$\begin{cases} -\Delta v + \nabla\mathfrak{q} = -\lambda v, & \text{in } G_+, \\ \nabla \operatorname{div} v = 0, & \text{in } G_+, \\ \gamma(v) = \phi := (0, \phi'), \end{cases}$$

with data $\lambda v \in \mathrm{L}^2(G_+)^n \cap \mathrm{L}^q_\omega(G_+)^n$ and $\phi \in \mathcal{S}(H)^n$, Lemma 5.23 gives $(v, \mathfrak{q}) \in W^{2,q}_\omega(G_+)^n \times \widehat{W}^{1,q}_\omega(G_+)$. As $|\lambda| = 1$, Theorem 5.24 and estimate (5.24) give

$$\|v, \nabla^2 v, \nabla\mathfrak{q}\|_{\mathrm{L}^q_\omega(G_+)} \le c(\|v\|_{\mathrm{L}^q_\omega(G_+)} + |\phi'|_{\widehat{T}^{2,q}_\omega(H)}) \le c\|\phi'\|_{T^{2,q}_\omega(H)}, \tag{5.25}$$

where $c = c(n, q, \omega, \vartheta) > 0$ is A_q-consistent. Clearly, this estimate extends to arbitrary $\phi' \in T^{2,q}_\omega(H)^{n-1}$ by density.

Combining the two problems solved on the whole group G and on G_+, respectively, we define $u := \bar{u}|_{G_+} + v$ and $\mathfrak{p} := \bar{\mathfrak{p}}|_{G_+} + \mathfrak{q}$. Then $(u, \mathfrak{p}) \in W^{2,q}_\omega(G_+)^n \times \widehat{W}^{1,q}_\omega(G_+)$ is a solution to the resolvent problem (1.2) with a corresponding A_q-consistent estimate.

Step 4: Uniqueness

Uniqueness follows by a standard duality argument based on the existence part, which is already proven. We omit the details here.

Step 5: Additional Regularity

From the previous steps it follows that $(u,\mathfrak{p}) \in W^{2,q_1}_{\omega_1}(G_+)^n \times \widehat{W}^{1,q_1}_{\omega_1}(G_+)$ can be written in the form $u = \bar{u}|_{G_+} + v$ and $\mathfrak{p} = \bar{\mathfrak{p}}|_{G_+} + \mathfrak{q}$. By the regularity assertion in Theorem 4.14, we obtain $(\bar{u},\bar{\mathfrak{p}}) \in W^{2,q_2}_{\omega_2}(G)^n \times \widehat{W}^{1,q_2}_{\omega_2}(G)$ and thus also $\phi' = \gamma(\bar{u}|_{G_+}) \in T^{2,q_2}_{\omega_2}(H)^{n-1}$.

By Corollary 5.18 we find a sequence $\{\phi'^k\}_{k\in\mathbb{N}} \subset C_0^\infty(H)^{n-1}$ such that $\phi'^k \to \phi'$ in $T^{2,q_1}_{\omega_1}(H)^{n-1} \cap T^{2,q_2}_{\omega_2}(H)^{n-1}$. Denote by $(v^k,\mathfrak{q}^k)$ the functions defined by (5.23) with ϕ'^k in place of ϕ'. Then Step 3 of this proof shows

$$(v^k,\mathfrak{q}^k) \in \left(W^{2,q_1}_{\omega_1}(G_+)^n \times \widehat{W}^{1,q_1}_{\omega_1}(G_+)\right) \cap \left(W^{2,q_2}_{\omega_2}(G_+)^n \times \widehat{W}^{1,q_2}_{\omega_2}(G_+)\right),$$

and the existence of a constant $c = c_1(n,q_1,\omega_1) + c_2(n,q_2,\omega_2) > 0$ such that

$$\|v^k,\nabla^2 v^k,\nabla\mathfrak{q}^k\|_{\mathrm{L}^{q_1}_{\omega_1}(G_+)\cap\mathrm{L}^{q_2}_{\omega_2}(G_+)} \le c\|\phi'^k\|_{T^{2,q_1}_{\omega_1}(H)\cap T^{2,q_2}_{\omega_2}(H)}.$$

Since $\{\phi'_k\}$ is a Cauchy sequence in $T^{2,q_1}_{\omega_1}(H)^{n-1} \cap T^{2,q_2}_{\omega_2}(H)^{n-1}$, so are v^k in $W^{2,q_2}_{\omega_2}(G_+)^n \cap W^{2,q_2}_{\omega_2}(G_+)^n$ and $\mathfrak{q}^k$ in $\widehat{W}^{1,q_1}_{\omega_1}(G_+) \cap \widehat{W}^{1,q_2}_{\omega_2}(G_+)$. Denote the limit by

$$(\tilde{v},\tilde{\mathfrak{q}}) \in \left(W^{2,q_1}_{\omega_1}(G_+)^n \times \widehat{W}^{1,q_1}_{\omega_1}(G_+)\right) \cap \left(W^{2,q_2}_{\omega_2}(G_+)^n \times \widehat{W}^{1,q_2}_{\omega_2}(G_+)\right).$$

By uniqueness in $W^{2,q_1}_{\omega_1}(G_+)^n \times \widehat{W}^{1,q_1}_{\omega_1}(G_+)$, we conclude $(v,\mathfrak{q}) = (\tilde{v},\tilde{\mathfrak{q}})$.

6. Bounded Domains

We begin with the Laplace resolvent problem.

Theorem 6.1. *Let $\Omega \subset G$ be a bounded domain of class $C^{1,1}$, $q, q_i \in (1,\infty)$, $\omega,\omega_i \in A_q(G)$, $i = 1,2$, $\vartheta \in (0,\pi)$ and $\lambda \in \Sigma_\vartheta \cup \{0\}$. For every $f \in \mathrm{L}^q_\omega(\Omega)$ there is a unique solution $u \in W^{2,q}_\omega(\Omega) \cap W^{1,q}_{0,\omega}(\Omega)$ to*

$$\begin{cases} \lambda u - \Delta u = f & \text{in } \Omega, \\ \qquad\quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

and there is an A_q-consistent $c = c(n,q,\omega,\vartheta,\Omega)) > 0$ such that

$$\|u,\lambda u,\nabla^2 u\|_{\mathrm{L}^q_\omega(G)} \le c\|f\|_{\mathrm{L}^q_\omega(G)}. \tag{6.2}$$

If $f \in \mathrm{L}^{q_2}_{\omega_2}(\Omega) \cap \mathrm{L}^{q_2}_{\omega_2}(\Omega)$, then the unique solution $u \in W^{2,q_1}_{\omega_1}(\Omega) \cap W^{1,q_1}_{0,\omega_1}(\Omega)$ also belongs to $u \in W^{2,q_2}_{\omega_2}(\Omega) \cap W^{1,q_2}_{0,\omega_2}(\Omega)$.

Proof. First of all, we observe that the assertion is true for the non-periodic case. This follows by a localization procedure as in [11, Theorem 3.3]. In the partially periodic case, we use a similar localization.

Namely, since Ω is bounded, for every δ there is a finite covering of $\overline{\Omega}$ by balls $B^1_\delta,\ldots,B^m_\delta \subset G$ with radius less than δ and non-negative cut-off functions $\psi_1,\ldots,\psi_m \in C_0^\infty(G)$ with $\operatorname{supp}\psi_j \subset B^j_\delta$ and $\sum_{j=1}^m \psi_j = 1$. It should be understood that we rule out superfluous base sets, *i.e.*, the case $B^j_\delta \cap \Omega = \emptyset$ does not occur. It is clear, that we can choose the base sets in such a way that $B^j_\delta \cap \partial\Omega = \emptyset$ implies $\overline{B^j_\delta} \subset \Omega$. We choose $\delta > 0$ so small, that each B^j_δ can be regarded as a ball in $\mathbb{R}^n$, that is $\delta < L/4$.

We first prove that if $u \in W^{2,q}_\omega(\Omega) \cap W^{1,q}_{0,\omega}(\Omega)$ is a solution to (6.1), then the estimate (6.2) holds with an additional term $\|u\|_{W^{1,q}_\omega(\Omega)}$ on the right-hand side. For $j \in \{1,\ldots,m\}$ it holds $\lambda(\psi_j u) - \Delta(\psi_j u) = f_j$ with $f_j := \psi_j f - 2(\nabla\psi_j)\nabla u - (\nabla\psi_j)u$, where we interpret these equations as problems in a bounded $C^{1,1}$-domains $\Omega_j \subset \mathbb{R}^n$, such that after identification of G with $\mathbb{R}^{n_1} \times [0,L)^{n_2}$ it holds $B^j_\delta \cap \Omega \subset \Omega_j$. Thus, it follows

$$\|\psi_j u,\lambda(\psi_j u),\nabla^2(\psi_j u)\|_{\mathrm{L}^q_\omega(\Omega_j)} \le c\|f_j\|_{\mathrm{L}^q_\omega(\Omega_j)} \le c(\psi_j)\left(\|f\|_{\mathrm{L}^q_\omega(\Omega_j)} + \|u\|_{W^{1,q}_\omega(\Omega_j)}\right). \tag{6.3}$$

Summing up the finitely many inequalities, the claim is proven.

Concerning uniqueness, testing the equation with u itself shows $u = 0$ for $q \geq 2$ and $\omega = 1$. For $q < 2$ and $\omega = 1$, we use the Sobolev embedding and (6.3) with $f = 0$ to see $u \in W^{2,s_1}(\Omega)$ for $\frac{1}{n} + \frac{1}{s_1} = \frac{1}{q}$. Repeating this procedure, we obtain $q < s_1 < \cdots < s_{k_q}$ with $s_{k_q} > 2$. Hence $u = 0$ also in this case. For general $\omega \in A_q(G)$, the problem can be reduced to the non-weighted situation in virtue of Lemma 3.3.

A compactness argument based on uniqueness as in the Lemma 3.7 below shows now, that the additional lower order term on the right-hand side may be omitted and hence the full *a priori* estimate (6.2) is valid. Therefore, the operator $(\lambda - \Delta)_{q,\omega} : W^{2,q}_\omega(\Omega) \cap W^{1,q}_{0,\omega}(\Omega) \to \mathrm{L}^q_\omega(\Omega)$ is closed.

By Riesz' representation theorem, we obtain for every $f \in \mathrm{L}^2(\Omega)$ a solution $u \in W^{1,2}_0(\Omega)$ with $\Delta u \in \mathrm{L}^q_\omega(\Omega)$ to (6.1). Using the partition of unity again, we obtain even $u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$. If $f \in \mathrm{L}^q(\Omega) \cap \mathrm{L}^2(\Omega)$, then $u \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$. Indeed, for $1 < q < 2$, this follows by the trivial embedding, whereas for $q > 2$, the regularity proof of the uniqueness assertion (with interchanged rôles of q and 2) applies. Therefore, by Lemma 3.3, the range of $(\lambda - \Delta)_{q,\omega}$ is dense. Since $(\lambda - \Delta)_{q,\omega}$ is also closed and injective, it is an isomorphism.

The regularity assertion is a consequence of the fact that there is a number $s \in (1, \infty)$ such that $\mathrm{L}^{q_1}_{\omega_1}(\Omega) + \mathrm{L}^{q_2}_{\omega_2}(\Omega) \subset \mathrm{L}^s(\Omega)$ and the uniqueness assertion on $\mathrm{L}^s(\Omega)$. □

Let us now turn to the Stokes resolvent problem. We consider the operator $S^\lambda_{q,\omega} : X^q_\omega \to Y^q_\omega$, where

$$X^q_\omega := \left(W^{2,q}_\omega(\Omega) \cap W^{1,q}_{0,\omega}(\Omega)\right)^n \times \widehat{W}^{1,q}_\omega(\Omega),$$
$$Y^q_\omega := \mathrm{L}^q_\omega(\Omega)^n \times \left(W^{1,q}_\omega(\Omega) \cap \widehat{W}^{-1,q}_{0,\omega}(\Omega)\right),$$

defined via

$$S^\lambda_{q,\omega}(u, \mathfrak{p}) := (\lambda u - \Delta u + \nabla \mathfrak{p}, -\operatorname{div} u).$$

Lemma 6.2. *Let $\Omega \subset G$ be a bounded domain of class $C^{1,1}$, $q \in (1, \infty)$, $\omega \in A_q(G)$, $\vartheta \in (0, \pi)$ and $\lambda \in \Sigma_\vartheta \cup \{0\}$. Assume $(u, \mathfrak{p}) \in X^q_\omega$ and define $(f, -g) := S^\lambda_{q,\omega}(u, \mathfrak{p})$.*

(i) *There exists an A_q-consistent $c = c(q, \omega, \theta, \Omega) > 0$ such that*

$$\|u, \lambda u, \nabla^2 u, \nabla \mathfrak{p}\|_{\mathrm{L}^q_\omega(\Omega)} \leq c \left(\|f, \nabla g\|_{\mathrm{L}^q_\omega(\Omega)} + |\lambda| \|g\|_{\widehat{W}^{-1,q}_{0,\omega}(\Omega)} + \|u, \mathfrak{p}\|_{\mathrm{L}^q_\omega(\Omega)} \right). \tag{6.4}$$

(ii) *The operator $S^\lambda_{q,\omega}$ is injective.*

Proof. Consider the partition of unity $\{\psi_j\}$, $j \in \{1, \ldots, m\}$ from the proof of Theorem 6.1. Since $(f, -g) = S^\lambda_{q,\omega}(u, \mathfrak{p})$, we obtain for $j \in \{1, \ldots, m\}$

$$\begin{aligned} \lambda(\psi_j u) - \Delta(\psi_j u) + \nabla(\psi_j \mathfrak{p}) &= f_j, \\ \operatorname{div}(\psi_j u) &= g_j, \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} f_j &:= \psi_j f - 2(\nabla \psi_j)\nabla u - (\Delta \psi_j) u + (\nabla \psi_j)\mathfrak{p}, \\ g_j &:= \psi_j g + (\nabla \psi_j) \cdot u. \end{aligned} \tag{6.6}$$

(i) We learn from [11, Theorem 3.3] on bounded domains of class $C^{1,1}$ in $\mathbb{R}^n$, that

$$\|\psi_j u, \lambda \psi_j u, \nabla^2(\psi_j u), \nabla(\psi_j \mathfrak{p})\|_{\mathrm{L}^q_\omega(\Omega_j)} \leq c \left(\|f_j, \nabla g_j\|_{\mathrm{L}^q_\omega(\Omega_j)} + \|\lambda g_j\|_{\widehat{W}^{-1,q}_{0,\omega}(\Omega_j)} \right).$$

By Corollary 3.5, the definition of f_j and g_j yields

$$\|f_j\|_{\mathrm{L}^q_\omega(\Omega_j)} \leq C(\psi_j) \left(\|f\|_{\mathrm{L}^q_\omega(\Omega_j)} + \|u\|_{W^{1,q}_\omega(\Omega_j)} + \|\mathfrak{p}\|_{\mathrm{L}^q_\omega(\Omega_j)} \right),$$
$$\|\nabla g_j\|_{\mathrm{L}^q_\omega(\Omega_j)} \leq C(\psi_j) \left(\|\nabla g\|_{\mathrm{L}^q_\omega(\Omega_j)} + \|u\|_{W^{1,q}_\omega(\Omega_j)} \right).$$

We still have to estimate the term $\|\lambda g_j\|_{\widehat{W}^{-1,q}_{0,\omega}(\Omega_j)}$. Let $v \in \widehat{W}^{1,q'}_{\omega'}(\Omega_j)$ and define $v_0 := v - \frac{1}{\mu(U)}\int_U v \,\mathrm{d}\mu$, where $U \subset \Omega_j$ is a bounded Lipschitz domain containing $\operatorname{supp} \nabla\psi_j \cap \Omega_j$. As v_0 has vanishing mean in U, Corollary 3.5 gives A_q-consistent constants $c_1, c_2 > 0$ such that

$$\begin{aligned}\|\nabla(\psi_j v_0)\|_{\mathrm{L}^{q'}_{\omega'}(\Omega)} &\le C_1(\psi_j)\left(\|v_0\|_{\mathrm{L}^{q'}_{\omega'}(U)} + \|\nabla v_0\|_{\mathrm{L}^{q'}_{\omega'}(\Omega_j)}\right)\\ &\le c_1\|\nabla v_0\|_{\mathrm{L}^{q'}_{\omega'}(\Omega_j)} = c_1\|\nabla v\|_{\mathrm{L}^{q'}_{\omega'}(\Omega_j)},\end{aligned}$$

and by a similar computation $\|(\nabla\psi_j)v_0\|_{W^{1,q'}_{\omega'}(\Omega)} \le c_2\|\nabla v\|_{\mathrm{L}^{q'}_{\omega'}(\Omega_j)}$. Note that

$$[g_j, v] = [\operatorname{div}(\psi_j u), v] = -[\psi_j u, \nabla v_0] = -[u, \nabla(\psi_j v_0)] + [u, (\nabla\psi_j)v_0].$$

Therefore, we can compute

$$\begin{aligned}\|g_j\|_{\widehat{W}^{-1,q}_{0,\omega}(\Omega_j)} &= \sup_{0\neq v\in\widehat{W}^{1,q'}_{\omega'}(\Omega_j)} \frac{|[g_j,v]|}{\|\nabla v\|_{\mathrm{L}^{q'}_{\omega'}(\Omega_j)}}\\ &\le c_1 \sup_{0\neq v\in\widehat{W}^{1,q'}_{\omega'}(\Omega)} \frac{|[u,\nabla v]|}{\|\nabla v\|_{\mathrm{L}^{q'}_{\omega'}(\Omega)}} + c_2\|u\|_{W^{-1,q}_{0,\omega}(\Omega)}\\ &= c_1\|g\|_{\widehat{W}^{-1,q}_{0,\omega}(\Omega)} + c_2\|u\|_{W^{-1,q}_{0,\omega}(\Omega)}.\end{aligned} \tag{6.7}$$

It follows

$$\|\lambda g_j\|_{\widehat{W}^{-1,q}_{0,\omega}(\Omega_j)} \le c\left(\|\lambda g\|_{\widehat{W}^{-1,q}_{0,\omega}(\Omega)} + \|\lambda u\|_{W^{-1,q}_{0,\omega}(\Omega)}\right).$$

Since we can estimate u by $\nabla^2 u$ in view of the *a priori* estimate in Lemma 3.6, summing up the finitely many inequalities obtained for $j \in \{1,\dots,m\}$ yields estimate (6.4), only with the additional terms $\|\nabla u\|_{\mathrm{L}^p_\omega(\Omega)}$ and $\|\lambda u\|_{W^{-1,q}_{0,\omega}(\Omega)}$ on the right-hand side. The norm of ∇u can be dealt with by Ehrling's lemma, absorbing the second-order part into the left-hand side. Moreover, since $\lambda u = \Delta u + f - \nabla\mathfrak{p}$, we have

$$\|\lambda u\|_{W^{-1,q}_{0,\omega}(\Omega)} \le \|\mathfrak{p}, \Delta u\|_{W^{-1,q}_{0,\omega}(\Omega)} + \|f\|_{\mathrm{L}^q_\omega(\Omega)}.$$

By Lemma 3.3 there is $s > 1$ such that $u \in W^{2,s}(\Omega)$. Recall that for every $0 < \alpha < \beta < 1$ and $r \in (1,\infty)$, the real interpolation space $(\mathrm{L}^r(\Omega), D(\Delta_r))_{\beta,r}$ is contained in the domain of the fractional Laplace operator $(-\Delta)^\alpha_r$, where $D(\Delta_r) := W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega)$. For adjoint fractional operators it holds $(A^\alpha)' = (A')^\alpha$, see [16, Corollary 5.2.4], and since $-\Delta'_s = -\Delta_{s'}$ by symmetry, we obtain $((-\Delta)^\alpha_s)' = (-\Delta)^\alpha_{s'}$. Fix $\alpha \in (0, 1/2s')$. Then $C^\infty_0(\overline{\Omega}) \subset \left(\mathrm{L}^{s'}(\Omega), D(\Delta_{s'})\right)_{\beta,s'} \subset D(\Delta^\alpha_{s'})$ for $\beta \in (\alpha, 1/2s')$, since the trace information of $\left(\mathrm{L}^{s'}(\Omega), D(\Delta_{s'})\right)_{\beta,s'}$ gets lost during the interpolation due to $2\beta < 1/s'$, see Lemma 4.1 in [2]. Hence,

$$|[\Delta u, \varphi]| = |[(-\Delta)^{1-\alpha}u, (-\Delta)^\alpha\varphi]| \le \|(-\Delta)^{1-\alpha}u\|_{\mathrm{L}^q_\omega(\Omega)}\|\varphi\|_{W^{1,q'}_{\omega'}(\Omega)}, \quad \varphi \in C^\infty_0(\overline{\Omega}),$$

which yields $\|\Delta u\|_{W^{-1,q}_{0,\omega}(\Omega)} \le \|(-\Delta)^{1-\alpha}u\|_{\mathrm{L}^q_\omega(\Omega)}$. By a standard result on fractional operators it holds $\|(-\Delta)^{1-\alpha}u\|_{\mathrm{L}^q_\omega(\Omega)} \le \varepsilon^{1-\alpha}\|u\|_{W^{2,q}_\omega(\Omega)} + C\varepsilon^{-\alpha}\|u\|_{\mathrm{L}^q_\omega(\Omega)}$ for every $\varepsilon > 0$, where C depends on α and the norm of $\{\lambda(\lambda-\Delta)^{-1} \mid \lambda > 0\}$ in $\mathcal{L}(\mathrm{L}^q_\omega(\Omega))$, which is finite and A_q-consistent by Theorem 6.1. Note that also α can be chosen A_q-consistently due to Lemma 3.3. Consequently, the constant C is A_q-consistent.

Similarly, we can show $\|\nabla\mathfrak{p}\|_{W^{-1,q}_{0,\omega}(\Omega)} \le \varepsilon^{1-2\alpha}\|\nabla\mathfrak{p}\|_{\mathrm{L}^q_\omega(\Omega)} + C\varepsilon^{-2\alpha}\|\mathfrak{p}\|_{\mathrm{L}^q_\omega(\Omega)}$. Choosing ε so small that we can absorb the higher norm terms into the left-hand side, we obtain the full estimate (6.4).

(ii) Assume $(f,g) = (0,0)$. Then for $\omega = 1$ and $q \ge 2$ it follows immediately $u = 0$ and hence $\nabla\mathfrak{p} = 0$ by testing the equation with u. Hence, an analogous argument as in the proof of Theorem 6.1 shows the uniqueness for all $q \in (1,\infty)$ and $\omega \in A_q(G)$.

$\square$

We will apply a compactness argument in order to show that the last term in (6.4) can be omitted.

Lemma 6.3. *Let $\Omega \subset G$ be a bounded periodic $C^{1,1}$-domain, $q \in (1,\infty)$, $\omega \in A_q(G)$, $\vartheta \in (0,\pi)$ and $\lambda \in \Sigma_\vartheta \cup \{0\}$. Assume $(u,\mathfrak{p}) \in X^q_\omega$ and define $(f,-g) := S^\lambda_{q,\omega}(u,\mathfrak{p})$. Then*

$$\|u, \lambda u, \nabla^2 u, \nabla \mathfrak{p}\|_{\mathrm{L}^q_\omega(\Omega)} \leq c\left(\|f, \nabla g\|_{\mathrm{L}^q_\omega(\Omega)} + |\lambda|\|g\|_{\widehat{W}^{-1,q}_{0,\omega}(\Omega)}\right), \tag{6.8}$$

where $c = c(q,\omega,\vartheta,\Omega) > 0$ is an A_q-consistent constant.

Proof. Assume the lemma was wrong. Then there is $R > 0$, a sequence $\{\omega_j\} \subset A_q(G)$ with upper bound $\sup_j \mathcal{A}_q(\omega_j) \leq R$, sequences $\{(u_j,\mathfrak{p}_j)\} \subset X^q_{\omega_j}(\Omega)$ and resolvent parameters $\{\lambda_j\} \subset \Sigma_\vartheta \cup \{0\}$ such that

$$\|u_j, \lambda_j u_j, \nabla^2 u_j, \nabla \mathfrak{p}_j\|_{L^q_{\omega_j}(\Omega)} > j\left(\|f_j, \nabla g_j\|_{L^q_{\omega_j}(\Omega)} + |\lambda_j|\|g_j\|_{\widehat{W}^{-1,q}_{0,\omega_j}(\Omega)}\right), \tag{6.9}$$

for all $j \in \mathbb{N}$, where we have set $(f_j,-g_j) := S^{\lambda_j}_{q,\omega_j}(u_j,\mathfrak{p}_j)$. Since the pressures are defined only up to a constant, we may assume that for all $j \in \mathbb{N}$ we have $\int_\Omega \mathfrak{p}_j \,\mathrm{d}x = 0$. Also, we can assume that the resolvent parameters λ_j converge to some $\lambda \in \overline{\Sigma_\vartheta} \cup \{\infty\}$.

Let Q be a bounded Lipschitz domain containing Ω. Then for $\omega_j := \omega_j/\omega_j(Q)$ both $\omega(Q) = 1$ and $\mathcal{A}_q(\omega_j) = \mathcal{A}_q(\omega_j) < R$ hold for all $j \in \mathbb{N}$. If we multiply (6.9) by $\omega_j(Q)^{-1/q}$, we obtain the same inequality with ω_j replaced by ω_j. In the following, we will suppress the notation ω_j and always write ω_j. By normalizing (6.9), we can assume

$$\begin{aligned} (\forall j \in \mathbb{N}) \quad \|u_j, \lambda_j u_j, \nabla^2 u_j, \nabla \mathfrak{p}_j\|_{\mathrm{L}^q_{\omega_j}(\Omega)} = 1, \\ \|f_j, \nabla g_j\|_{L^q_{\omega_j}(\Omega)} + |\lambda_j|\|g_j\|_{\widehat{W}^{-1,q}_{0,\omega_j}(\Omega)} \to 0, \quad \text{as } j \to \infty. \end{aligned} \tag{6.10}$$

The assertion about the uniformity in Lemma 3.3 shows that there is an $s > 1$ such that $L^q_{\omega_j}(\Omega) \hookrightarrow L^s(\Omega)$ with an embedding constant independent of $j \in \mathbb{N}$. Hence, the sequences $\{\lambda_j u_j\}$, $\{u_j\}$ and $\{\nabla \mathfrak{p}_j\}$ are bounded in $L^s(\Omega)^n$, $W^{2,s}(\Omega)^n$ and $L^s(\Omega)^n$, respectively. Taking subsequences if necessary, we thus obtain the weak convergences

$$\lambda_j u_j \rightharpoonup v, \quad u_j \rightharpoonup u, \quad \nabla \mathfrak{p}_j \rightharpoonup \nabla \mathfrak{p}, \tag{6.11}$$

in their respective spaces. Again, we can assume $\int_\Omega \mathfrak{p} \,\mathrm{d}x = 0$. The weak convergence $u_j \rightharpoonup u$ in $W^{2,s}(\Omega)$ gives $u_j \to u$ in $W^{1,s}(\Omega)$ by the compact embedding and hence $\gamma(u) = 0$ due to $\gamma(u_j) = 0$. This shows $\int_\Omega \operatorname{div} u = 0$, and since $\nabla \operatorname{div} u = 0$ by the convergence $\nabla g_j \to 0$ in $\mathrm{L}^s(\Omega)$, even $\operatorname{div} u = 0$. The convergence $\lambda_j g_j \to 0$ in the sense of distributions yields $(v, \nabla\varphi) = \lim_{j\to\infty}(\lambda_j u_j, \nabla\varphi) = -\lim_{j\to\infty}(\lambda_j g_j, \varphi) = 0$ for $\varphi \in C_0^\infty(\overline{\Omega})$, which shows both $\operatorname{div} v = 0$ and $v \cdot n|_{\partial\Omega} = 0$. Hence,

$$\begin{cases} v - \Delta u + \nabla \mathfrak{p} = 0, & \text{in } \Omega, \\ \operatorname{div} v = \operatorname{div} u = 0, & \text{in } \Omega, \\ v \cdot n|_{\partial\Omega} = 0. \end{cases} \tag{6.12}$$

We proceed by distinguishing the cases $\lambda \neq \infty$ and $\lambda = \infty$. Assume first $\lambda_j \to \lambda \neq \infty$. Then (6.11) implies $\lambda u = v$. Moreover, λ is still contained in a sector $\Sigma_{\vartheta'} \cup \{0\}$. Hence, by (6.12) it follows $S^\lambda_s(u,\mathfrak{p}) = (0,0)$ and by injectivity of S^λ_s we obtain $u = 0$ and $\nabla \mathfrak{p} = 0$. Therefore, Lemma 3.7 shows $\|\mathfrak{p}_j\|_{L^q_{\omega_j}(\Omega)} \to 0$ and $\|u_j\|_{\mathrm{L}^q_{\omega_j}(\Omega)} \to 0$. By Lemma 6.2 we know

$$\|u_j, \lambda u_j, \nabla^2 u_j, \nabla \mathfrak{p}_j\|_{L^q_{\omega_j}(\Omega)} \leq c\left(\|f_j, \nabla g_j\|_{L^q_{\omega_j}(\Omega)} + |\lambda_j|\|g_j\|_{\widehat{W}^{-1,q}_{0,\omega_j}(\Omega)} + \|u_j, \mathfrak{p}_j\|_{\mathrm{L}^q_{\omega_j}(\Omega)}\right),$$

where the constant $c > 0$ is independent of $j \in \mathbb{N}$, since it is A_q-consistent and $\mathcal{A}_q(\omega_j) \leq R$. Sending $j \to \infty$, we obtain the contradiction $1 \leq 0$.

In the case $\lambda_j \to \infty$, we necessarily have $\|u_j\|_{L^q_{\omega_j}(\Omega)} \to 0$ due to (6.10). Then (6.12) is the trivial Helmholtz decomposition and thus $v = \nabla \mathfrak{p} = 0$. By the same arguments as in the case $\lambda \neq \infty$, we obtain a contradiction. ∎

6.1. Proof of Theorem 1.4 in Bounded Domains

It is left to show that the range of the operator $S^\lambda_{q,\omega}$ is dense in $Y^q_\omega(\Omega)$. Let us introduce a restricted operator $S^\lambda_{q,\sigma}: X^q_\sigma(\Omega) \to \mathrm{L}^q(\Omega)^n$ via $S^\lambda_{q,\sigma}(u,\mathfrak{p}) = \lambda u - \Delta u + \nabla\mathfrak{p}$. Here, $X^q_\sigma(\Omega) := \{(u,\mathfrak{p}) \in X^q(\Omega) : \operatorname{div} u = 0\}$. As a first auxiliary result we want to show that the range of $S^\lambda_{q,\sigma}$ is dense in $\mathrm{L}^q(\Omega)^n$. Observe that for $q = 2$, every $f \in \mathrm{L}^2(\Omega)$ has a unique decomposition $(\lambda - \Delta)u + \nabla\mathfrak{p} = f$ in the sense of distributions with $\mathfrak{p} \in \widehat{W}^{1,2}(\Omega)$ and $u \in W^{1,2}_0(\Omega)^n$ with $\Delta u \in \mathrm{L}^2(\Omega)^n$ and $\operatorname{div} u = 0$. Indeed, this decomposition follows easily from the Helmholtz decomposition and the Riesz representation theorem. Let us check that $(u,\mathfrak{p}) \in X^2_\sigma(\Omega)$, using the partition of unity. Then for f_j and g_j, where we set $g = 0$ in the definition of g_j, it holds $(f_j, g_j) \in Y^2(\Omega_j)$ and hence $(\psi_j u, \psi_j\mathfrak{p}) \in X^2(\Omega_j)$. It follows $(u,\mathfrak{p}) \in X^2(\Omega)$ and consequently, since $\operatorname{div} u = 0$, even $(u,\mathfrak{p}) \in X^2_\sigma(\Omega)$. This shows that $S^\lambda_{2,\sigma}: X^2_\sigma(\Omega) \to \mathrm{L}^2(\Omega)^n$ is surjective.

If $q \in (1,\infty)$, let $f \in \mathrm{L}^q(\Omega)^n \cap \mathrm{L}^2(\Omega)^n$. By what we have just proven, there is a unique $(u,\mathfrak{p}) \in X^2_\sigma(\Omega)$ such that $S^\lambda_{2,\sigma}(u,\mathfrak{p}) = f$. Again by the same argumentation as above, using the partition of unity, we obtain $(u,\mathfrak{p}) \in X^q_\sigma(\Omega)$. Hence, since $\mathrm{L}^q(\Omega)^n \cap \mathrm{L}^2(\Omega)^n$ is dense in $\mathrm{L}^q(\Omega)^n$, the operator $S^\lambda_{q,\sigma}: X^q_\sigma(\Omega) \to \mathrm{L}^q(\Omega)^n$ has a dense range. Since the range is also closed by the *a priori* estimate (6.8), $S^\lambda_{q,\sigma}$ is even an isomorphism.

Let now $(f,g) \in Y^q(\Omega)$. Then in complete analogy to Lemma 5.5 in [9], we obtain $v \in (W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega))^n$ with $\operatorname{div} v = g$. Moreover, thanks to the fact that $S^\lambda_{q,\sigma}$ is an isomorphism, we can define $(w,\mathfrak{p}) \in X^q_\sigma(\Omega)$ as the preimage of $f - (\lambda v - \Delta v)$. Then $(u,\mathfrak{p}) := (v+w,\mathfrak{p}) \in X^q(\Omega)$ and $S^\lambda_q(u,\mathfrak{p}) = (f,-g)$. Therefore, the assertion is proven for $q \in (1,\infty)$ and $\omega = 1$.

If $\omega \in A_q(G)$ is arbitrary, we employ Lemma 3.3 to obtain $1 < r < \infty$ such that $\mathrm{L}^r(\Omega) \hookrightarrow \mathrm{L}^q_\omega(\Omega)$. Then $S^\lambda_{q,\omega} = S^\lambda_r$ on $X^r(\Omega)$. Since the range of S^λ_r is dense in $Y^r(\Omega)$ and $Y^r(\Omega)$ is itself dense in $Y^q_\omega(\Omega)$, we obtain a complete proof of the first part.

The additional regularity assertion follows as in Theorem 6.1 from $\mathrm{L}^{q_1}_{\omega_1}(\Omega) + \mathrm{L}^{q_2}_{\omega_2}(\Omega) \subset L^s(\Omega)$.

7. Appendix: Helmholtz Decomposition

In this appendix we want to establish the weighted Helmholtz decomposition in the case of $\Omega = G, G_+$ or a bounded periodic $C^{1,1}$-domain. Let us define for $q \in (1,\infty)$ and $\omega \in A_q(G)$ the spaces

$$\begin{aligned} C^\infty_{0,\sigma}(\Omega) &:= \{u \in C^\infty_0(\Omega)^n : \operatorname{div} u = 0\}, \\ \mathrm{L}^q_{\omega,\sigma}(\Omega) &:= \overline{C^\infty_{0,\sigma}(\Omega)^n}^{\|\cdot\|_{\mathrm{L}^q_\omega(\Omega)}}, \\ \nabla\widehat{W}^{1,q}_\omega(\Omega) &:= \{\nabla\mathfrak{p} : \mathfrak{p} \in \widehat{W}^{1,q}_\omega(\Omega)\}, \\ X^q_{\omega,\sigma}(\Omega) &:= \{u \in \mathrm{L}^q_\omega(\Omega)^n : \operatorname{div} u = 0,\ u \cdot n|_{\partial\Omega} = 0\}, \end{aligned}$$

where the norms of $\nabla\widehat{W}^{1,q}_\omega(\Omega)$ and $X^q_{\omega,\sigma}(\Omega)$ are given by $\|\nabla\mathfrak{p}\|_{\mathrm{L}^q_\omega(\Omega)}$ and $\|u\|_{\mathrm{L}^q_\omega(\Omega)}$, respectively. Observe that $\nabla\widehat{W}^{1,q}_\omega(\Omega)$ and $X^q_{\omega,\sigma}(\Omega)$ are closed subspaces of $\mathrm{L}^q_\omega(\Omega)^n$ and hence Banach spaces. The normal trace $u \cdot n|_{\partial\Omega}$ is well-defined by Stokes' theorem due to $\operatorname{div} u = 0$.

Lemma 7.1. *Let $\Omega = G_+$ or let Ω be a bounded periodic $C^{1,1}$-domain. Assume $q, q_i \in (1,\infty)$ and $\omega \in A_q(G)$, $\omega_i \in A_{q_i}(G)$, $i = 1,2$, respectively, $\vartheta \in (0,\pi)$ and $\lambda \in \Sigma_\vartheta$.*

(i) *For all $F \in \widehat{W}^{-1,q}_{0,\omega}(\Omega)$ there exists a unique solution $u \in \widehat{W}^{1,q}_\omega(\Omega)$ to*

$$(\nabla u, \nabla\varphi) = [F,\varphi], \quad \varphi \in \widehat{W}^{1,q'}_{\omega'}(\Omega), \tag{7.1}$$

and there is an A_q-consistent $c = c(n,q,\omega,\Omega) > 0$ such that

$$\|\nabla u\|_{\mathrm{L}^q_\omega(\Omega)} \le c\|F\|_{\widehat{W}^{-1,q}_{0,\omega}(\Omega)}. \tag{7.2}$$

If $F \in \widehat{W}^{-1,q_1}_{0,\omega_1}(\Omega) \cap \widehat{W}^{-1,q_2}_{0,\omega_2}(\Omega)$*, then the unique solution* $u \in \widehat{W}^{1,q_1}_{\omega_1}(\Omega)$ *to* (7.1) *satisfies* $u \in \widehat{W}^{1,q_1}_{\omega_1}(\Omega) \cap \widehat{W}^{1,q_2}_{\omega_2}(\Omega)$.

(ii) *For all* $F \in \widehat{W}^{-1,q}_{0,\omega}(\Omega)$ *there exists a unique solution* $u \in W^{1,q}_{\omega}(\Omega) \cap \widehat{W}^{-1,q}_{\omega}(\Omega)$ *to*

$$\lambda(u,\varphi) + (\nabla u, \nabla\varphi) = [F,\varphi], \quad \varphi \in \widehat{W}^{1,q'}_{\omega'}(\Omega),$$

and there is an A_q*-consistent* $c = c(n,q,\omega,\Omega) > 0$ *such that*

$$\lambda\|u\|_{\widehat{W}^{-1,q}_{\omega}(\Omega)} + \|\nabla u\|_{\mathrm{L}^q_\omega(\Omega)} \le c\|F\|_{\widehat{W}^{-1,q}_{0,\omega}(\Omega)}.$$

Proof. (i) For $\Omega = G$, this is just Proposition 4.7. If $\Omega = G_+$, we can assume $\omega = \omega^*$ by Lemma 3.2, and hence the assertion follows from the whole space result by a reflection argument, if one defines $f \in \widehat{W}^{-1,q}_{\omega}(G)$ via $[f,\psi] := [F,\varphi]$ for $\psi \in \widehat{W}^{1,q'}_{\omega'}(G)$ with $\varphi := (\psi + \psi^*)|_{G_+} \in \widehat{W}^{1,q'}_{\omega'}(G_+)$.

Let Ω be a bounded periodic $C^{1,1}$-domain now and assume that for $u \in \widehat{W}^{1,q}_{\omega}(\Omega)$ solves (7.1) with $F = 0$. If $q \ge 2$ and $\omega = 1$, it immediately follows $|\nabla u|^2 = 0$. For $1 < q < 2$ and $\omega = 1$, we use a localization as in Sect. 6 to obtain $u \in \widehat{W}^{1,2}(\Omega)$. In the presence of a weight $\omega \in A_q(G)$, Lemma 3.3 yields $1 < s < \infty$ such that $u \in \widehat{W}^{1,q}_{\omega}(\Omega) \hookrightarrow \widehat{W}^{1,s}(\Omega)$, and hence the uniqueness follows also in this case.

Moreover, in the unweighted case $\omega = 1$, we obtain a solution to (7.1) for all $F \in \widehat{W}^{-1,q}_0(\Omega) \cap \widehat{W}^{-1,2}_0(\Omega)$. Indeed, there is a solution $u \in \widehat{W}^{1,2}(\Omega)$ by the Riesz representation theorem. Therefore, using again the partition of unity, we obtain as above (with interchanged rôles of q and 2) the result $u \in \widehat{W}^{1,q}(\Omega)$.

If $\omega \in A_q(G)$ is arbitrary, we find in virtue of Lemma 3.3 a number $q \le r < \infty$ such that $\widehat{W}^{1,r}(\Omega) \hookrightarrow \widehat{W}^{1,q}_{\omega}(\Omega)$. Hence, in any case we obtain a dense subset of $D \subset \widehat{W}^{-1,q}_{0,\omega}(\Omega)$ such that for every $F \in D$ there is a solution $u \in \widehat{W}^{1,q}_{\omega}(\Omega)$ to (7.1). The partition of unity shows that for $u \in \widehat{W}^{1,q}_{\omega}(\Omega)$ we have the estimate

$$\|\nabla u\|_{\mathrm{L}^q_\omega(\Omega)} \le c\left(\|F\|_{\widehat{W}^{-1,q}_{0,\omega}(\Omega)} + \|u\|_{\mathrm{L}^q_\omega(\Omega)}\right),$$

where $c = c(n,q,\omega,\Omega) > 0$ is A_q-consistent. By the same compactness argument as in the proof of Lemma 6.3, we can improve this estimate to the full *a priori* estimate (7.2), and hence $D = \widehat{W}^{-1,q}_{0,\omega}(\Omega)$.

The regularity assertion is a consequence of the fact that there is a number $1 < s < \infty$ such that $\widehat{W}^{1,q_1}_{\omega_1}(\Omega) + \widehat{W}^{1,q_2}_{\omega_2}(\Omega) \subset \widehat{W}^{1,s}(\Omega)$ and the uniqueness assertion on $\widehat{W}^{1,s}(\Omega)$.

(ii) Analogous. Observe that for $\Omega = G$, this is Proposition 4.9(iii).

□

Theorem 7.2. *Let* $q, q_i \in (1,\infty)$*,* $\omega \in A_q(G)$ *and* $\omega_i \in A_{q_i}(G)$*,* $i = 1,2$.

(i) *The following algebraic and topological decomposition holds*

$$\mathrm{L}^q_\omega(\Omega)^n = X^q_{\omega,\sigma}(\Omega) \oplus \nabla\widehat{W}^{1,q}_{\omega}(\Omega).$$

This decomposition is A_q*-consistent,* i.e., *for the corresponding* Helmholtz projection operator $P_{q,\omega} : \mathrm{L}^q_\omega(\Omega)^n \to X^q_{\omega,\sigma}(\Omega)$ *with kernel* $\nabla\widehat{W}^{1,q}_{\omega}(\Omega)$ *it holds*

$$\|P_{q,\omega}u\|_{X^q_{\omega,\sigma}(\Omega)} \le c\|u\|_{\mathrm{L}^q_\omega(\Omega)},$$

where $c = c(n,q,\omega,\Omega) > 0$ *is* A_q*-consistent.*

(ii) $\mathrm{L}^q_{\omega,\sigma}(\Omega) = X^q_{\omega,\sigma}(\Omega)$.

(iii) *The dual space* $(\mathrm{L}^q_{\omega,\sigma}(\Omega))'$ *can be identified with* $L^{q'}_{\omega',\sigma}(\Omega)$ *and we have* $(P_{q,\omega})' = P_{q',\omega'}$.

(iv) *If* $u \in \mathrm{L}^{q_1}_{\omega_1}(\Omega)^n \cap \mathrm{L}^{q_2}_{\omega_2}(\Omega)^n$*, then* $P_{q_1,\omega_1}u = P_{q_2,\omega_2}u$.

Proof. We show the assertion for $\Omega = G$ only, the other cases following analogously.

(i) Let $u \in \mathrm{L}^q_\omega(G)^n$. By Lemma 7.1 there exists a unique $\mathfrak{p} \in \widehat{W}^{1,q}_\omega(G)$ such that $(\nabla\mathfrak{p}, \nabla\varphi) = (u, \nabla\varphi)$, $\varphi \in \widehat{W}^{1,q'}_{\omega'}(G)$, and we have

$$\|\nabla\mathfrak{p}\|_{\mathrm{L}^q_\omega(G)} \le c\|u\|_{\mathrm{L}^q_\omega(G)},$$

where $c = c(n, q, \omega) > 0$ is A_q-consistent. Thus, $P_{q,\omega}u := u - \nabla\mathfrak{p}$ is well-defined and it is clear from the construction that $P_{q,\omega} : \mathrm{L}^q_\omega(G)^n \to X^q_{\omega,\sigma}(G)$ is an A_q-consistently bounded, surjective and linear projection with kernel $\nabla\widehat{W}^{1,q}_\omega(G)$.

(ii) Since the inclusion $\mathrm{L}^q_{\omega,\sigma}(G) \subset X^q_{\omega,\sigma}(G)$ and the norm equality are trivial, it suffices to show that $C^\infty_{0,\sigma}(G)$ is dense in $X^q_{\omega,\sigma}(G)$. Let us first show that the dual space $(X^q_{\omega,\sigma}(G))'$ can be identified with $X^{q'}_{\omega',\sigma}(G)$. The embedding $X^{q'}_{\omega',\sigma}(G) \subset (X^q_{\omega,\sigma}(G))'$ follows by Hölder's inequality. Conversely, let $\psi \in (X^q_{\omega,\sigma}(G))'$. The theorem of Hahn–Banach provides a $v \in \mathrm{L}^{q'}_{\omega'}(G)^n$ such that $[\psi, w] = (v, w)$ for all $w \in X^q_{\omega,\sigma}(G)$. We employ the Helmholtz decomposition from part (i) to receive $v = P_{q',\omega'}v + \nabla\mathfrak{p}_v$ with $\mathfrak{p}_v \in \widehat{W}^{1,q'}_{\omega'}(G)$ and $P_{q',\omega'}v \in X^{q'}_{\omega',\sigma}(G)$ and hence $[\psi, w] = (P_{q',\omega'}v, w)$ for all $w \in X^q_{\omega,\sigma}(G)$. Thus, $\psi \in (X^q_{\omega,\sigma}(G))'$ can be identified with $P_{q',\omega'}v \in X^{q'}_{\omega',\sigma}(G)$.

Let $\psi \in X^{q'}_{\omega',\sigma}(G) = (X^q_{\omega,\sigma}(G))'$ be such that $[\psi, \varphi] = 0$ for all $C^\infty_{0,\sigma}(G)$. Let us use the notation $\tilde{\Omega} := \mathbb{R}^{n_1} \times [0, L)^{n_2}$. Then by the canonical identification of $\tilde{\Omega}$ and G we obtain $C^\infty_{0,\sigma}(\tilde{\Omega}) \subset C^\infty_{0,\sigma}(G)$. Therefore, $[\psi, \varphi] = 0$ for all $\varphi \in C^\infty_{0,\sigma}(\tilde{\Omega})$, where we view ψ as a distribution on $\tilde{\Omega}$. By de Rham's argument [7], there is a distribution $\mathfrak{p}$ such that

$$\nabla\mathfrak{p} = \psi \in \mathrm{L}^{q'}_{\omega'}(\tilde{\Omega}). \tag{7.3}$$

It follows immediately that $\mathfrak{p} \in \widehat{W}^{1,q'}_{\omega'}(\tilde{\Omega})$, but if $n_2 \ge 1$, we need to show the periodicity assertion $\mathfrak{p} \in \widehat{W}^{1,q'}_{\omega'}(G)$. Hence, assume for now $n_2 = 1$, so that $y = x_n$ and $n_1 = n - 1$. It suffices to show that $\mathfrak{q}(x') := \mathfrak{p}(x', L) - \mathfrak{p}(x', 0) = 0$ for all $x' \in \mathbb{R}^{n-1}$. Let $\varphi_n \in C^\infty_0(\mathbb{R}^{n-1})$ be arbitrary and define $\varphi := (0, \dots, 0, \varphi_n) \in C^\infty_{0,\sigma}(G)$. Then (7.3) gives

$$0 = [\psi, \varphi] = \int_{\mathbb{R}^{n-1}} \int_0^L \partial_n\mathfrak{p}\, \mathrm{d}x_n\, \varphi_n\, \mathrm{d}x' = (\mathfrak{q}, \varphi_n)_{\mathbb{R}^{n-1}},$$

which shows $\mathfrak{q} = 0$. Therefore, we can extend $\mathfrak{p}$ periodically with respect to the variable x_n to $\mathfrak{p} \in \widehat{W}^{1,q'}_{\omega'}(G)$. If $n_2 \ge 1$, an analogous argument yields the same assertion.

It follows $[\psi, v] = (\nabla\mathfrak{p}, v) = 0$ for all $v \in X^q_{\omega,\sigma}(G)$ and hence $\psi = 0$ in $(X^q_{\omega,\sigma}(G))'$. Consequently, $C^\infty_{0,\sigma}(G)$ is dense in $X^q_{\omega,\sigma}(G)$.

(iii) The assertion $(\mathrm{L}^q_{\omega,\sigma}(G))' = L^{q'}_{\omega',\sigma}(G)$ has already been proven in part (ii). Moreover,

$$\begin{aligned} [u, (P_{q,\omega})'v] &= (P_{q,\omega}u, v) = (P_{q,\omega}u, P_{q',\omega'}v + \nabla\mathfrak{p}_v) = (P_{q,\omega}u, P_{q',\omega'}v) \\ &= (P_{q,\omega}u + \nabla\mathfrak{p}_u, P_{q',\omega'}v) = (u, P_{q',\omega'}v), \end{aligned}$$

whenever $u \in \mathrm{L}^q_\omega(G)$ and $v \in \mathrm{L}^{q'}_{\omega'}(G)$. Consequently $(P_{q,\omega})' = P_{q',\omega'}$.

(iv) Lemma 7.1 yields $\nabla\mathfrak{p}_1 = \nabla\mathfrak{p}_2$ and hence $P_{q_1,\omega_1}u = P_{q_2,\omega_2}u$.

□

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